10. Autonomous Linear Dynamical Systems

- Linear dynamical systems
- Phase plane
- RLC circuits
- Chemical reactions
- Markov chains
- Higher-order linear systems
- Mechanical systems
- Linearization

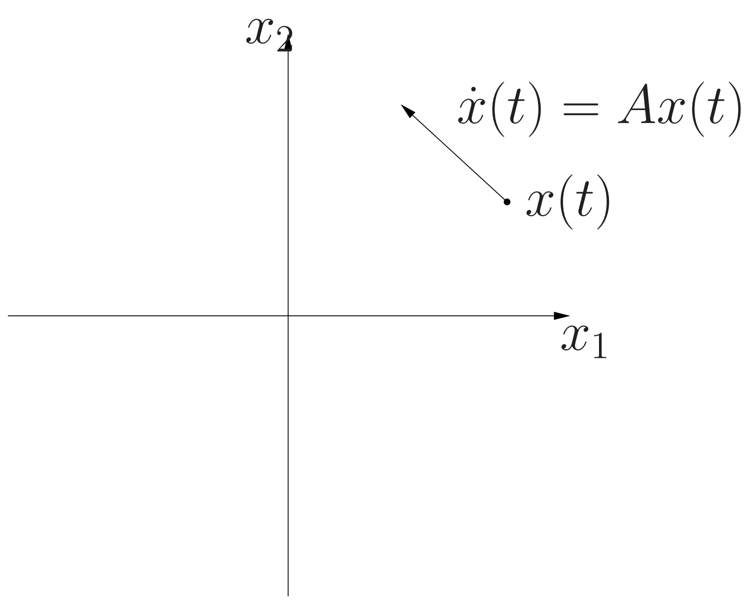
Autonomous linear dynamical systems

continuous-time autonomous LDS has form

$$\dot{x} = Ax$$

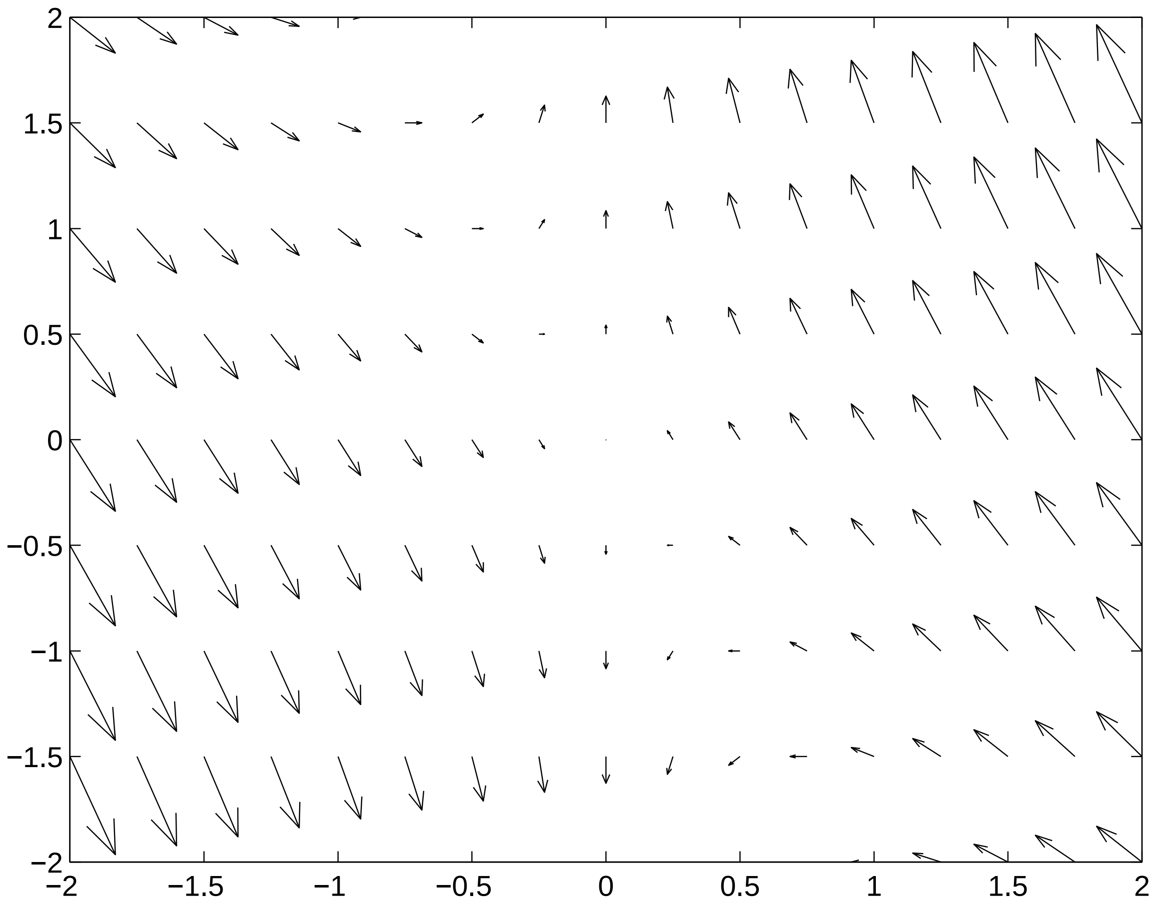
- $x(t) \in \mathbb{R}^n$ is called the state
- n is the *state dimension* or (informally) the *number of states*
- A is the *dynamics matrix*
  (system is *time-invariant* if A doesn't depend on t)

picture (*phase plane*):



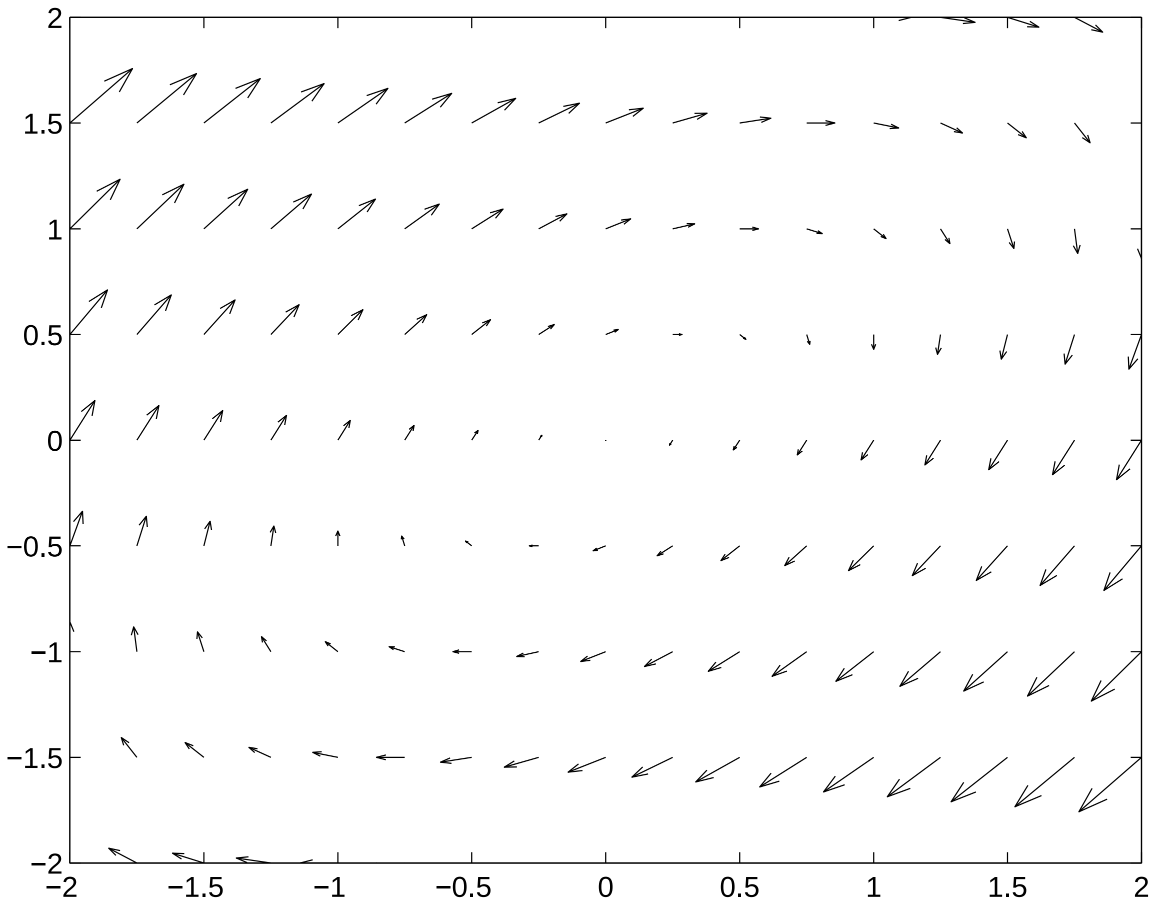
Example 1: Autonomous linear dynamical systems

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} x$$

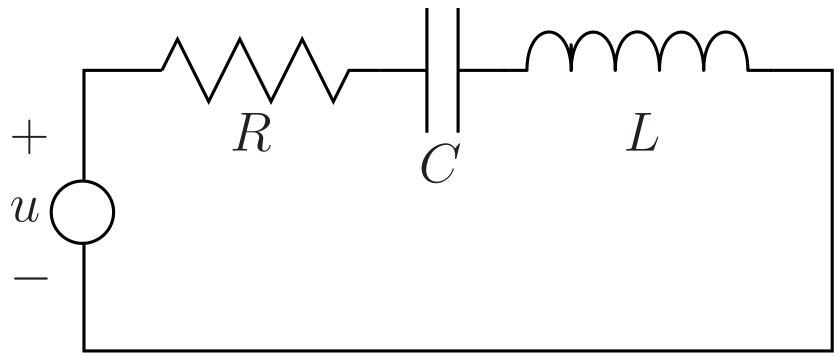


Example 2: Autonomous linear dynamical systems

$$\dot{x} = \begin{bmatrix} -0.5 & 1 \\ -1 & 0.5 \end{bmatrix} x$$



Linear circuit



Let x_1 be the voltage across C and x_2 be the current through L, then

$$\begin{aligned} u &= Rx_2 + x_1 + L\dot{x}_2 \\ x_2 &= C\dot{x}_1 \end{aligned}$$

and hence $\dot{x} = Ax + Bu$ where

$$A = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1/L \end{bmatrix}$$

Chemical reactions

- reaction involving n chemicals; x_i is concentration of chemical i
- linear model of reaction kinetics

$$\frac{dx_i}{dt} = a_{i1}x_1 + \cdots + a_{in}x_n$$

Example: series reaction

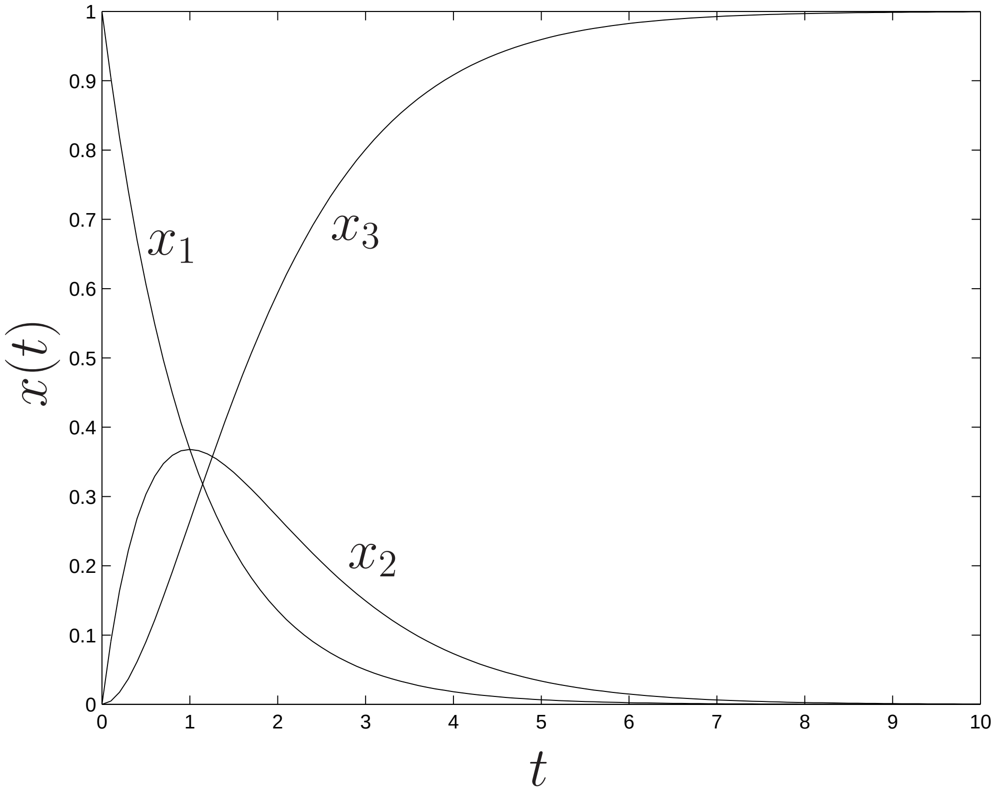
$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

with linear dynamics

$$\dot{x} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} x$$

plot for $k_1 = k_2 = 1$

initial $x(0) = (1, 0, 0)$



Finite-state discrete-time Markov chain

$z(t) \in \{1, \ldots, n\}$ is a random sequence with

$$\mathrm{Prob}(\ z(t+1) = i \mid z(t) = j\ ) = P_{ij}$$

where $P \in \mathbb{R}^{n \times n}$ is the matrix of *transition probabilities*

represent probability distribution of $z(t)$ as n-vector

$$p(t) = \begin{bmatrix} \mathrm{Prob}(z(t) = 1) \\ \vdots \\ \mathrm{Prob}(z(t) = n) \end{bmatrix}$$

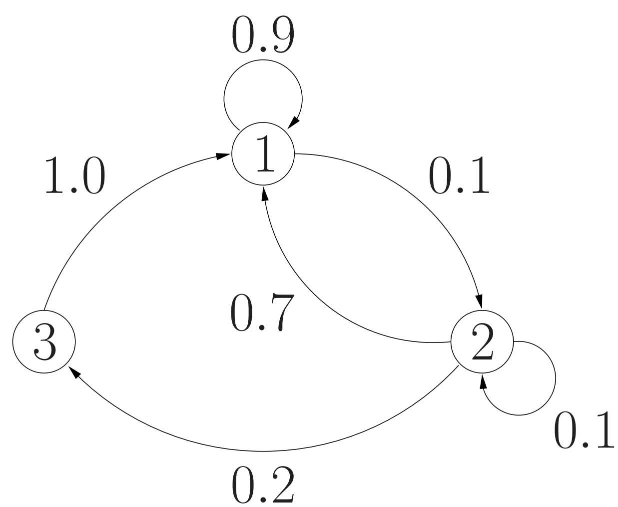
then

$$\boxed{p(t+1) = Pp(t)}$$

Example: Finite-state Markov chain

Markov chain is depicted graphically

- nodes are states
- edges show transition probabilities



- state 1 is 'system OK'
- state 2 is 'system down'
- state 3 is 'system being repaired'

$$p(t+1) = \begin{bmatrix} 0.9 & 0.7 & 1.0 \\ 0.1 & 0.1 & 0 \\ 0 & 0.2 & 0 \end{bmatrix} p(t)$$

Higher order linear dynamical systems

$$x^{(k)} = A_{k-1}x^{(k-1)} + \cdots + A_1 x^{(1)} + A_0 x, \quad x(t) \in \mathbb{R}^n$$

where $x^{(m)}$ denotes mth derivative

define new variable $z = \begin{bmatrix} x \\ x^{(1)} \\ \vdots \\ x^{(k-1)} \end{bmatrix} \in \mathbb{R}^{nk}$, so

$$\dot{z} = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(k)} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & I \\ A_0 & A_1 & A_2 & \cdots & A_{k-1} \end{bmatrix} z$$

a (first order) LDS (with bigger state)

Mechanical systems

mechanical system with k degrees of freedom undergoing small motions:

$$M\ddot{q} + D\dot{q} + Kq = 0$$

- $q(t) \in \mathbb{R}^k$ is the vector of generalized displacements
- M is the *mass matrix*
- K is the *stiffness matrix*
- D is the *damping matrix*

with state $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$ we have

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x$$

Linearization near equilibrium point

nonlinear, time-invariant differential equation (DE):

$$\dot{x} = f(x)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$

suppose x_e is an *equilibrium point*, *i.e.*, $f(x_e) = 0$
(so $x(t) = x_e$ satisfies DE)

now suppose $x(t)$ is near x_e, so

$$\dot{x}(t) = f(x(t)) \approx f(x_e) + Df(x_e)(x(t) - x_e)$$

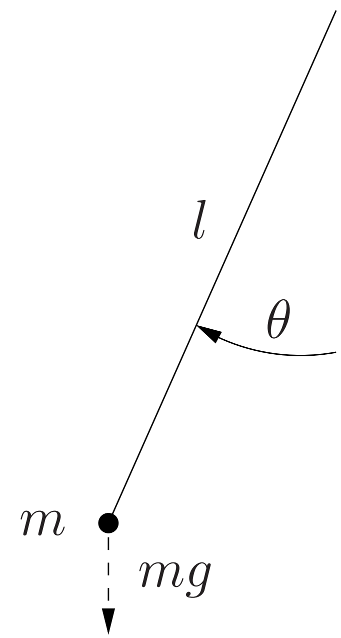
with $\delta x(t) = x(t) - x_e$, rewrite as

$$\dot{\delta x}(t) \approx Df(x_e)\delta x(t)$$

replacing $\approx$ with $=$ yields *linearized approximation* of DE near x_e

we *hope* solution of $\dot{\delta x} = Df(x_e)\delta x$ is a good approximation of $x - x_e$

Example: pendulum



2nd order nonlinear DE $ml^2\ddot{\theta} = -lmg\sin\theta$

rewrite as first order DE with state $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$:

$$\dot{x} = \begin{bmatrix} x_2 \\ -(g/l)\sin x_1 \end{bmatrix}$$

equilibrium point (pendulum down): $x = 0$

linearized system near $x_e = 0$:

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ -g/l & 0 \end{bmatrix} \delta x$$

Does linearization 'work'?

the linearized system usually, but not always, gives a good idea of the system behavior near x_e

example 1: $\dot{x} = -x^3$ near $x_e = 0$

for $x(0) > 0$ solutions have form $x(t) = \left(x(0)^{-2} + 2t\right)^{-1/2}$

linearized system is $\dot{\delta x} = 0$; solutions are constant

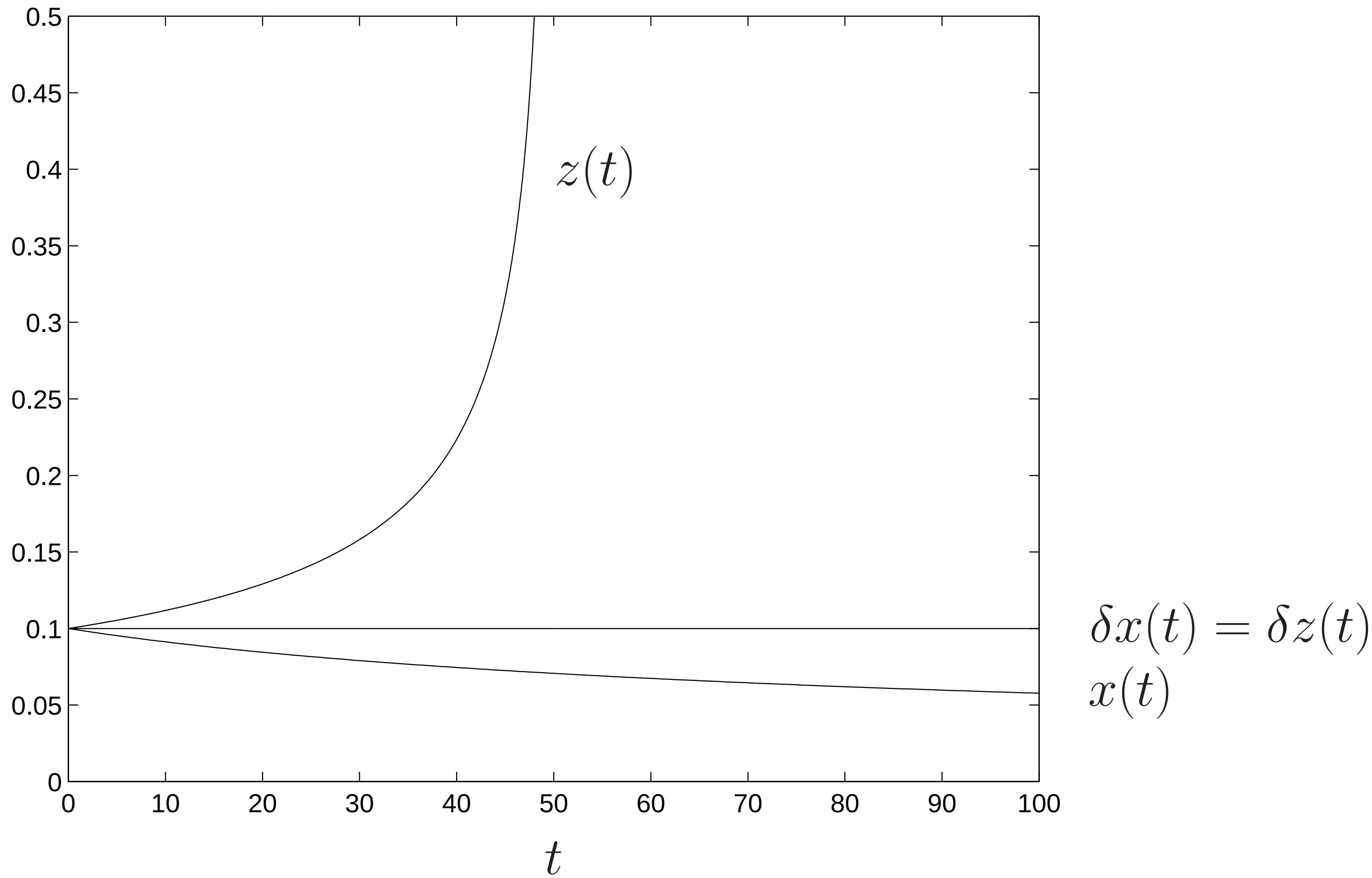
example 2: $\dot{z} = z^3$ near $z_e = 0$

for $z(0) > 0$ solutions have form $z(t) = \left(z(0)^{-2} - 2t\right)^{-1/2}$

(finite escape time at $t = z(0)^{-2}/2$)

linearized system is $\dot{\delta z} = 0$; solutions are constant

Does linearization 'work'?



- systems with very different behavior have same linearized system
- linearized systems do not predict qualitative behavior of either system