

13. Controllability

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The Key Points of This Section

- we can compute minimum energy inputs so that $x(T) = x_{\text{des}}$
- we can measure controllability by looking at the SVD of the matrix

$$[A^{T-1}B \quad A^{T-2}B \quad \dots \quad AB \quad B]$$

for large T

- the singular values and left singular vectors give us the *controllability ellipsoid*, which tell us strong and weak directions in the state-space
- to compute this as $T \rightarrow \infty$, we solve the Lyapunov equation

$$W - AWA^T = BB^T$$

the eigenvectors of W are the axis directions and lengths of the controllability ellipsoid (over infinite time)

Controlling the State

discrete-time LDS, $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$

$$x(t+1) = Ax(t) + Bu(t) \quad x(0) = 0$$

look at state at time T

$$x(T) = \begin{bmatrix} A^{T-1}B & A^{T-2}B & \dots & AB & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(T-1) \end{bmatrix}$$

ask *control* questions:

- find input sequence $u(0), \dots, u(T-1)$ so that $x(T) = x_{\text{des}}$
- find all input sequences that result in $x(T) = x_{\text{des}}$
- among all those, find the smallest, most efficient one

how to control the state

$$x(T) = \begin{bmatrix} A^{T-1}B & A^{T-2}B & \dots & AB & B \end{bmatrix} \begin{bmatrix} u(0) \\ \vdots \\ u(T-1) \end{bmatrix} = H_T \begin{bmatrix} u(0) \\ \vdots \\ u(T-1) \end{bmatrix}$$

minimum norm solution is $\begin{bmatrix} u(0) \\ \vdots \\ u(T-1) \end{bmatrix} = H_T^\dagger x_{\text{des}}$

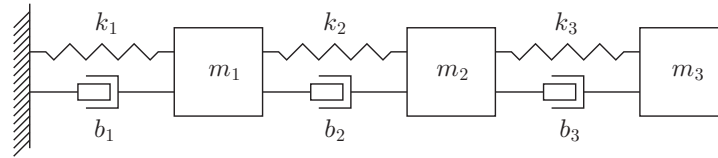
among all input sequences for which $x(T) = x_{\text{des}}$, this one has the smallest norm;

i.e, it minimizes

$$\sum_{t=0}^{T-1} \|u(t)\|^2 = \|u(0)\|^2 + \|u(1)\|^2 + \dots + \|u(T-1)\|^2$$

called the *input energy*

example: mass-spring system



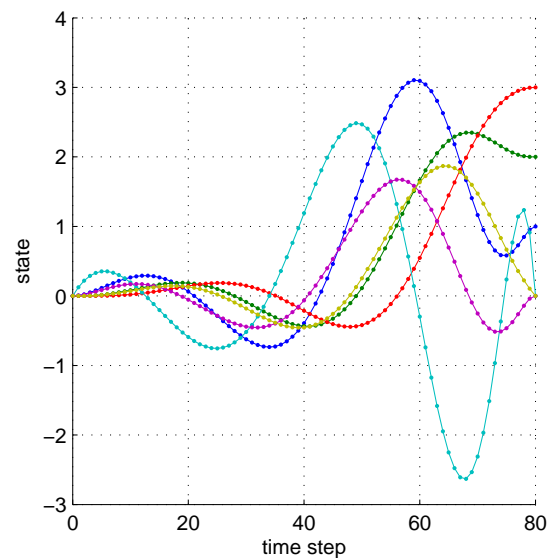
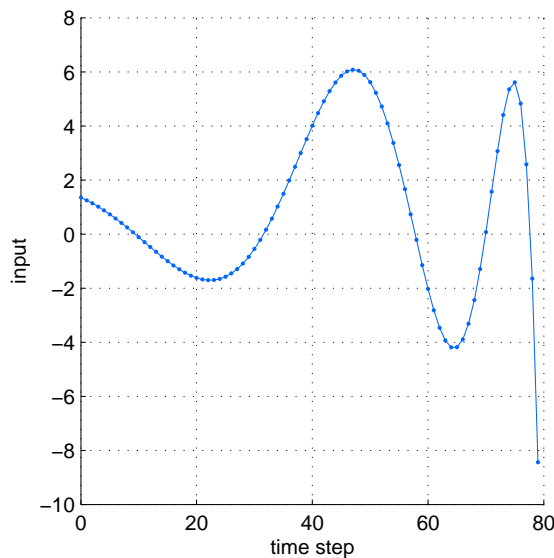
masses $m_i = 1$, spring constants $k = 1$, damping constants $b = 0.8$

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -1.6 & 0.8 & 0 \\ 1 & -2 & 1 & 0.8 & -1.6 & 0.8 \\ 0 & 1 & -1 & 0 & 0.8 & -0.8 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$u(t)$ is force applied to mass 1

$x_{\text{des}} = [1 \ 2 \ 3 \ 0 \ 0 \ 0]^T$ at time step $T = 80$.

example: control of spring-mass system



sampling period $h = 0.1$, optimal input achieves desired state

$$x(80) = x_{\text{des}} = [1 \ 2 \ 3 \ 0 \ 0 \ 0]^T$$

what next?

- what about assumptions? H_T is fat and full rank when?
- computation size; we need to compute H_T^\dagger
- how does solution behave as we change T ?

Reachable Set

- we need H_T to be fat

H_T has dimensions $n \times Tm$, so if $Tm < n$ we will have $\text{range}(H_T) \neq \mathbb{R}^n$

- we need H_t to be full rank

again, if not, then $\text{range}(H_T) \neq \mathbb{R}^n$

- $\text{range}(H_T)$ is the set of states which are *reachable* at time T
- $\text{range}(H_t) \subset \text{range}(H_s)$ if $t \leq s$, so we can reach more points given more time

because

$$H_T = [A^{T-1}B \quad A^{T-2}B \quad \dots \quad AB \quad B]$$

Cayley-Hamilton

characteristic polynomial

$$\begin{aligned} p(\lambda) &= \det(\lambda I - A) \\ &= a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 \end{aligned}$$

note that $a_n \neq 0$

Cayley-Hamilton theorem

A satisfies its own characteristic equation

that is

$$a_n A^n + a_{n-1} A^{n-1} + \dots + a_1 A + a_0 I = 0$$

important consequence: $A^n \in \text{span}\{I, A, A^2, \dots, A^{n-1}\}$

controllability

we have

$$H_T = [A^{T-1}B \quad A^{T-2}B \quad \dots \quad AB \quad B]$$

for $t \geq n$, we can express A^t as a linear combination of $A^{n-1}, A^{n-2}, \dots, A, I$

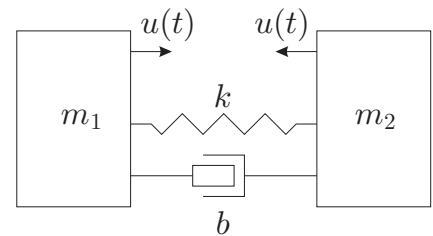
so if $T \geq n$, then

$$\text{range}(H_T) = \text{range}(H_n)$$

- if we are interested in reaching x_{des} in time $T \geq n$, we just need to look at H_n to determine the reachable set
- H_n is called the *controllability matrix*
- if $\text{range}(H_n) = \mathbb{R}^n$ the system is called *controllable* or *reachable*.
sometimes we just say (A, B) is reachable

example: controlling identical masses

- identical masses connected by spring and damper
- input is tension force between masses
- masses, spring, damping constants = 1



continuous-time system

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} u(t)$$

- starting from $x(0) = 0$, which states can we reach?

example: controlling identical masses

discretization $A_d = e^{Ah}$ $B_d = \int_0^h e^{As} B ds$

discrete-time controllability matrix $H_n = [A_d^3 B_d \quad A_d^2 B_d \quad A_d B_d \quad B_d]$

compute reachable set = $\text{range}(H_n)$ using SVD $H_n = U \Sigma V^T$

we find

$$\text{range}(H_n) = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

can reach those states with $x_1 = -x_2$ and $\dot{x}_1 = -\dot{x}_2$

because tension force does not change center of mass, or total momentum

energy required

pseudo-inverse

$$H_T^\dagger = H_T^T (H_T H_T^T)^{-1}$$

minimum norm input sequence achieving $x(T) = x_{\text{des}}$ is

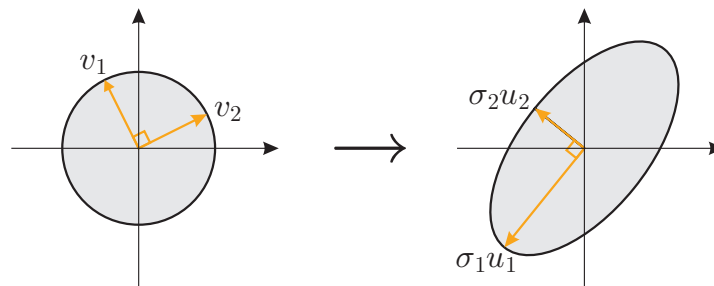
$$\begin{bmatrix} u(0) \\ \vdots \\ u(T-1) \end{bmatrix} = H_T^\dagger x_{\text{des}}$$

input energy required

$$\begin{aligned} \sum_{t=0}^T \|u(t)\|^2 &= x_{\text{des}}^T (H_T^\dagger)^T H_T^\dagger x_{\text{des}} \\ &= x_{\text{des}}^T (H_T H_T^T)^{-1} x_{\text{des}} \end{aligned}$$

controllability ellipsoid

look at the set of states reachable at time T with energy $\sum_{t=0}^T \|u(t)\|^2 \leq 1$



- *controllability ellipsoid*:
 semi-axis directions are left singular vectors of H_T
 semi-axis lengths are singular values of H_T
- short axis is *weakly controllable* direction
- *practical method* for determining controllability;
 gives a quantitative answer, not just yes or no

controllability ellipsoid

singular values and left singular vectors are eigenvalues and eigenvectors of $H_T H_T^T$

call this matrix W_T

$$W_T = [A^{T-1}B \ A^{T-2}B \ \dots \ AB \ B] [A^{T-1}B \ A^{T-2}B \ \dots \ AB \ B]^T$$

$$= \sum_{k=0}^{T-1} A^k B B^T (A^T)^k$$

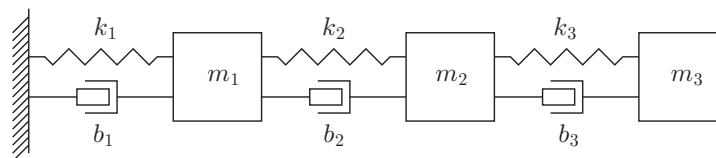
- energy required to reach x_{des} at time T is $x_{\text{des}}^T W_T^{-1} x_{\text{des}}$

- if $t \geq s$ then $W_t \geq W_s$ so $W_t^{-1} \leq W_s^{-1}$, so

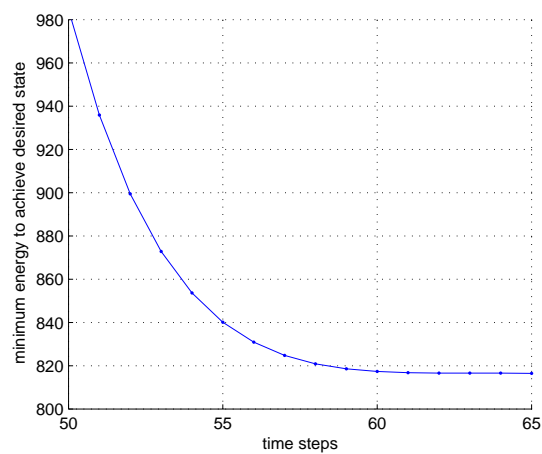
$$x_{\text{des}}^T W_t^{-1} x_{\text{des}} \leq x_{\text{des}}^T W_s^{-1} x_{\text{des}}$$

less energy is required to reach x_{des} given more time

example: energy against time



energy required



infinite time problems

what happens as $T \rightarrow \infty$?

let

$$\begin{aligned} W &= \lim_{T \rightarrow \infty} W_T \\ &= \lim_{T \rightarrow \infty} \sum_{k=0}^T A^k B B^T (A^T)^k \end{aligned}$$

this converges if $\rho(A) < 1$, that is the system is stable.

- W converges exponentially
- eigenvalues of W measure amount of energy required to reach corresponding eigenvectors with no restriction on how long it takes
- good practical measure of controllability
- how to compute it?

Lyapunov Equation

if A is stable, i.e., $\rho(A) < 1$, then

$$W = \sum_{k=0}^{\infty} A^k B B^T (A^T)^k$$

we have

$$\begin{aligned} A W A^T &= \sum_{k=1}^{\infty} A^k B B^T (A^T)^k \\ &= W - B B^T \end{aligned}$$

W satisfies the *Lyapunov equation*

$$W - A W A^T = B B^T$$

example: solving Lyapunov equations

they are just linear equations, so it's easy

$$A = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{4} \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

we want to solve $W - AW A^T = BB^T$, i.e.,

$$\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

write this as

$$\begin{bmatrix} w_{11} \\ w_{21} \\ w_{12} \\ w_{22} \end{bmatrix} - \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & \frac{1}{8} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{8} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{16} \end{bmatrix} \begin{bmatrix} w_{11} \\ w_{21} \\ w_{12} \\ w_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

which is just a set of linear equations in w_{ij}

solving Lyapunov equations

$$W = [w_1 \ \dots \ w_n] \quad BB^T = [b_1 \ \dots \ b_n]$$

we can write $W - AW A^T = BB^T$ as

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} - \begin{bmatrix} a_{11}A & a_{12}A & \dots & a_{1n}A \\ a_{21}A & & & \\ \vdots & & & \vdots \\ a_{n1}A & a_{n2}A & \dots & a_{nn}A \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

this is n^2 linear equations in n^2 unknowns

solutions of Lyapunov equations

we know that if

$$W = \sum_{k=0}^{\infty} A^k B B^T (A^T)^k$$

then W satisfies the equation

$$W - A W A^T = B B^T$$

key question: is there always a solution to this equation?

is it unique?

solutions of Lyapunov equations

let $Q = [q_1 \ \dots \ q_n]$ and $\hat{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$

write the Lyapunov equation $W - A W A^T = Q$ as

$$(I - \hat{A})\hat{w} = \hat{q}$$

if $\rho(A) < 1$ then for any $Q \in \mathbb{R}^{n \times n}$ we have a solution to $W - A W A^T = Q$ which is

$$W = \sum_{k=0}^{\infty} A^k Q (A^T)^k$$

so $\text{range}(I - \hat{A}) = \mathbb{R}^{n^2}$. i.e.,

if $\rho(A) < 1$ then $(I - \hat{A})$ is invertible

solutions of Lyapunov equations

so we have

if $\rho(A) < 1$ then the solution W to $W - AWA^T = BB^T$ is unique

we can solve this to determine *controllability*; it tells us

- the ellipsoid

$$\left\{ x \in \mathbb{R}^n \mid x^T W^{-1} x \leq 1 \right\}$$

is the set of states reachable with input energy $\sum_{t=0}^{\infty} \|u(t)\|^2 \leq 1$

- the corresponding eigenvectors of W tell us the strongly and weakly controllable directions