

5. Eigenvectors

- Eigenvectors and eigenvalues
- Properties
- Scaling interpretations
- Diagonalization

Eigenvectors and eigenvalues

$\lambda \in \mathbb{C}$ is an *eigenvalue* of $A \in \mathbb{C}^{n \times n}$ if

$$\mathcal{X}(\lambda) = \det(\lambda I - A) = 0$$

equivalent to:

- there exists nonzero $v \in \mathbb{C}^n$ s.t. $(\lambda I - A)v = 0$, i.e.,

$$Av = \lambda v$$

any such v is called an *eigenvector* of A (associated with eigenvalue λ)

- there exists nonzero $w \in \mathbb{C}^n$ s.t. $w^T(\lambda I - A) = 0$, i.e.,

$$w^T A = \lambda w^T$$

any such w is called a *left eigenvector* of A

Properties

- if v is an eigenvector of A with eigenvalue λ , then so is αv , for any $\alpha \in \mathbb{C}$, $\alpha \neq 0$
- even when A is real, eigenvalue λ and eigenvector v can be complex
- when A and λ are real, we can always find a real eigenvector v associated with λ : if $Av = \lambda v$, with $A \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{R}$, and $v \in \mathbb{C}^n$, then

$$A \operatorname{Re} v = \lambda \operatorname{Re} v, \quad A \operatorname{Im} v = \lambda \operatorname{Im} v$$

so $\operatorname{Re} v$ and $\operatorname{Im} v$ are real eigenvectors, if they are nonzero
(and at least one is)

- *conjugate symmetry*: if A is real and $v \in \mathbb{C}^n$ is an eigenvector associated with $\lambda \in \mathbb{C}$, then \bar{v} is an eigenvector associated with $\bar{\lambda}$:

taking conjugate of $Av = \lambda v$ we get $\overline{Av} = \overline{\lambda v}$, so

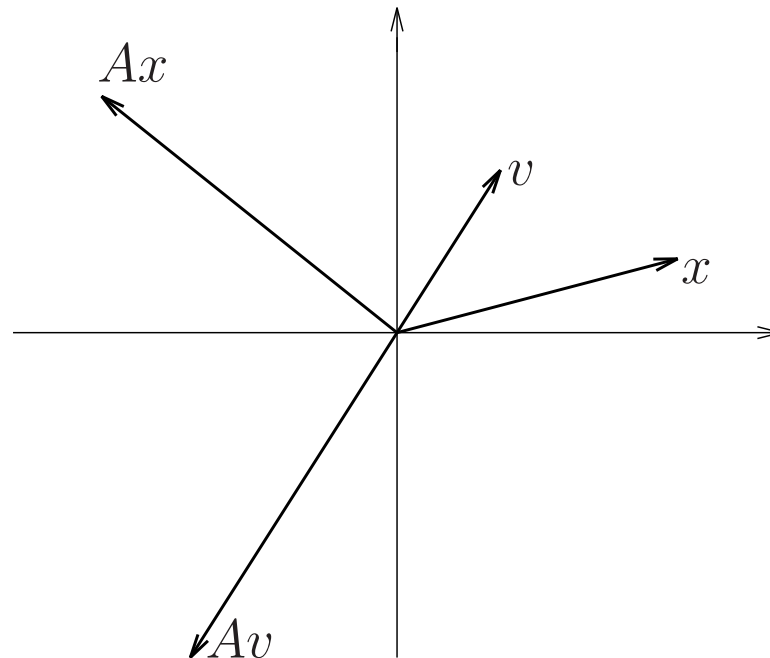
$$A\bar{v} = \bar{\lambda}\bar{v}$$

we'll assume A is real from now on ...

Scaling interpretation

(assume $\lambda \in \mathbb{R}$ for now; we'll consider $\lambda \in \mathbb{C}$ later)

if v is an eigenvector, effect of A on v is very simple: scaling by λ



(what is λ here?)

Properties

- $\lambda \in \mathbb{R}, \lambda > 0$: v and Av point in same direction
- $\lambda \in \mathbb{R}, \lambda < 0$: v and Av point in opposite directions
- $\lambda \in \mathbb{R}, |\lambda| < 1$: Av smaller than v
- $\lambda \in \mathbb{R}, |\lambda| > 1$: Av larger than v

(we'll see later how this relates to stability of continuous- and discrete-time systems...)

Diagonalization

suppose v_1, \dots, v_n is a *linearly independent* set of eigenvectors of $A \in \mathbb{R}^{n \times n}$:

$$Av_i = \lambda_i v_i, \quad i = 1, \dots, n$$

Write these equations as

$$AT = T\Lambda$$

Here $T = [v_1 \ \cdots \ v_n]$ and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ so that

$$A [v_1 \ \cdots \ v_n] = [v_1 \ \cdots \ v_n] \begin{bmatrix} \lambda_1 & & \\ & \cdots & \\ & & \lambda_n \end{bmatrix}$$

Diagonalization

We say A is *diagonalizable* if there exists T such that

$$T^{-1}AT \quad \text{is diagonal}$$

This is equivalent to A having a set of n linearly independent eigenvectors

- T invertible since v_1, \dots, v_n linearly independent
- similarity transformation by T *diagonalizes* A
- if A is not diagonalizable, it is called *defective*
- if A has distinct eigenvalues, *i.e.*, $\lambda_i \neq \lambda_j$ for $i \neq j$, then A is diagonalizable

Not all matrices are diagonalizable

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

characteristic polynomial is $\mathcal{X}(s) = s^2$, so $\lambda = 0$ is only eigenvalue

eigenvectors satisfy $Av = 0v = 0$, *i.e.*

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

so all eigenvectors have form $v = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$ where $v_1 \neq 0$

thus, A cannot have two independent eigenvectors

Diagonalization and left eigenvectors

rewrite $T^{-1}AT = \Lambda$ as $T^{-1}A = \Lambda T^{-1}$, or

$$\begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix} A = \Lambda \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}$$

where w_1^T, \dots, w_n^T are the rows of T^{-1}

thus

$$w_i^T A = \lambda_i w_i^T$$

i.e., the rows of T^{-1} are linearly independent left eigenvectors, normalized so that

$$w_i^T v_j = \delta_{ij}$$

left and right eigenvectors chosen this way are called *dual bases*