11. Matrix Exponential

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Diagonalization

suppose $v_1, \ldots, v_n$ is a *linearly independent* set of eigenvectors of $A \in \mathbb{R}^{n \times n}$:

$$\boxed{Av_i = \lambda_i v_i, \quad i = 1, \ldots, n}$$

express as

$$A \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

define $T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ and $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, so

$$\boxed{AT = T\Lambda}$$

- hence $T^{-1}AT = \Lambda$
- T invertible since $v_1, \ldots, v_n$ linearly independent
- similarity transformation by T *diagonalizes* A

Modal form

Suppose A is diagonalizable by T. Define new coordinates by $x = T\tilde{x}$, so

$$T\dot{\tilde{x}} = AT\tilde{x} \quad \Leftrightarrow \quad \dot{\tilde{x}} = T^{-1}AT\tilde{x}$$

in new coordinate system, system is diagonal

$$\boxed{\dot{\tilde{x}} = \Lambda\tilde{x}}$$

- Called *modal form*; trajectories consist of n independent modes, *i.e.*,

$$\tilde{x}_i(t) = e^{\lambda_i t}\tilde{x}_i(0)$$

- if initial state $x(0)$ is an eigenvector v, resulting motion is simple — always on the line spanned by v
- for $\lambda \in \mathbb{R}$, $\lambda < 0$, mode contracts or shrinks as $t \uparrow$
- for $\lambda \in \mathbb{R}$, $\lambda > 0$, mode expands or grows as $t \uparrow$

Matrix exponential

For a square matrix M, the **matrix exponential** is defined to be

$$\boxed{e^M = I + M + \frac{M^2}{2!} + \cdots}$$

Then the solution of of $\dot{x} = Ax$, with $A \in \mathbb{R}^{n\times n}$ is

$$\boxed{x(t) = e^{tA}x(0)}$$

- the series for e^M converges for all M
- one can check that $e^{tA}x(0)$ satisfies the ODE by differentiating
- natural generalization of scalar case

Properties of the Matrix exponential

- matrix exponential is *meant* to look like scalar exponential
- some things you'd guess hold for the matrix exponential (by analogy with the scalar exponential) do in fact hold
- but **many things you'd guess are wrong**

example: you might guess that $e^{A+B} = e^A e^B$, but it's false (in general)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$e^A = \begin{bmatrix} 0.54 & 0.84 \\ -0.84 & 0.54 \end{bmatrix}, \qquad e^B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$e^{A+B} = \begin{bmatrix} 0.16 & 1.40 \\ -0.70 & 0.16 \end{bmatrix} \neq e^A e^B = \begin{bmatrix} 0.54 & 1.38 \\ -0.84 & -0.30 \end{bmatrix}$$

Properties of the Matrix exponential

- $e^{A+B} = e^A e^B$ if $AB = BA$

- e^A is nonsingular, with inverse

$$\left(e^A\right)^{-1} = e^{-A}$$

  because

$$e^A e^{-A} = e^{A-A} = e^0 = I$$

- $\frac{d}{dt} e^{tA} = A e^{tA} = e^{tA} A$

- Matlab command: `expm(A)`, *not* `exp(A)`

Example: Matrix exponential

let's find e^A, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

We have

$$e^A = I + A + \frac{A^2}{2!} + \cdots = I + A$$

since $A^2 = A^3 = \cdots = 0$

Time transfer property

- For $\dot{x} = Ax$ we know

$$x(t) = e^{tA}x(0)$$

interpretation: the matrix e^{tA} propagates initial condition into state at time t

- More generally we have, for *any* t and h,

$$x(s + t) = e^{tA}x(s)$$

To see this, apply result above to $z(t) = x(t + s)$

interpretation: the matrix e^{tA} propagates state t seconds forward in time

Sampling a continuous-time system

suppose $\dot{x} = Ax$

sample x at times $t_1 \leq t_2 \leq \cdots$: define $z(k) = x(t_k)$

then $z(k+1) = e^{(t_{k+1}-t_k)A} z(k)$

for uniform sampling $t_{k+1} - t_k = h$, so

$$z(k+1) = e^{hA} z(k),$$

a discrete-time LDS (called *discretized version* of continuous-time system)

Complex eigenvectors

suppose $Av = \lambda v$, $v \neq 0$, λ is complex

for $a \in \mathbb{C}$, (complex) trajectory $ae^{\lambda t}v$ satisfies $\dot{x} = Ax$

hence so does (real) trajectory

$$\begin{aligned} x(t) &= \Re\left(ae^{\lambda t}v\right) \\ &= e^{\sigma t}\begin{bmatrix} v_{\rm re} & v_{\rm im} \end{bmatrix}\begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}\begin{bmatrix} \alpha \\ -\beta \end{bmatrix} \end{aligned}$$

where

$$v = v_{\rm re} + jv_{\rm im}, \quad \lambda = \sigma + j\omega, \quad a = \alpha + j\beta$$

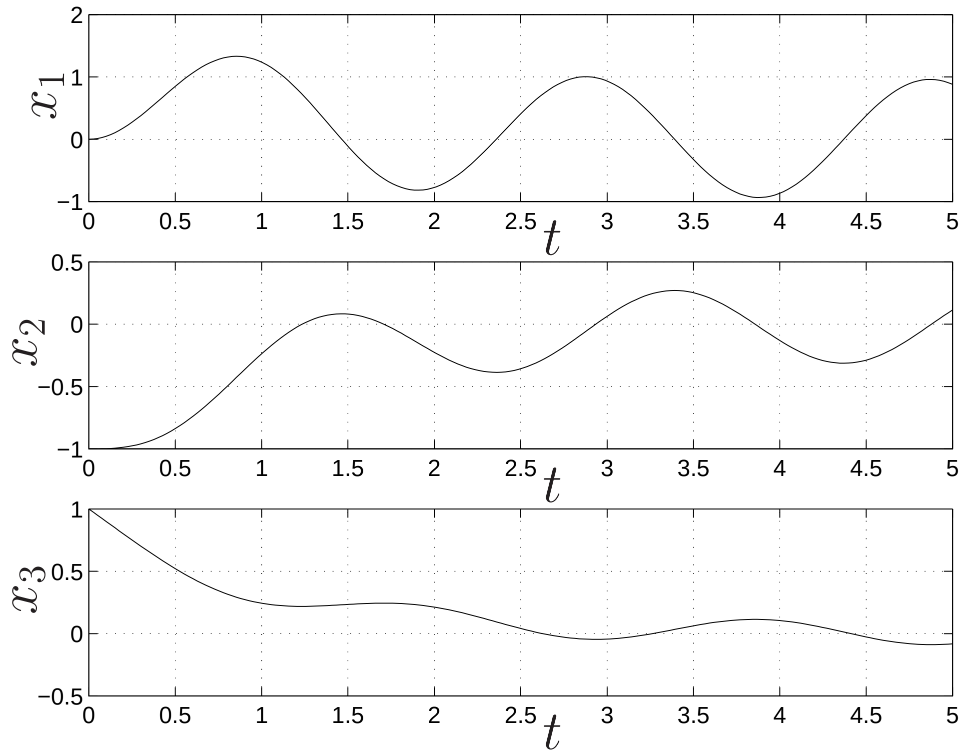
- trajectory stays in *invariant plane* $\operatorname{span}\{v_{\rm re}, v_{\rm im}\}$
- σ gives logarithmic growth/decay factor
- ω gives angular velocity of rotation in plane

Example 1

$$\dot{x} = \begin{bmatrix} -1 & -10 & -10 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x$$

eigenvalues are $-1, \ \pm j\sqrt{10}$

trajectory with $x(0) = (0, -1, 1)$:



Example 1

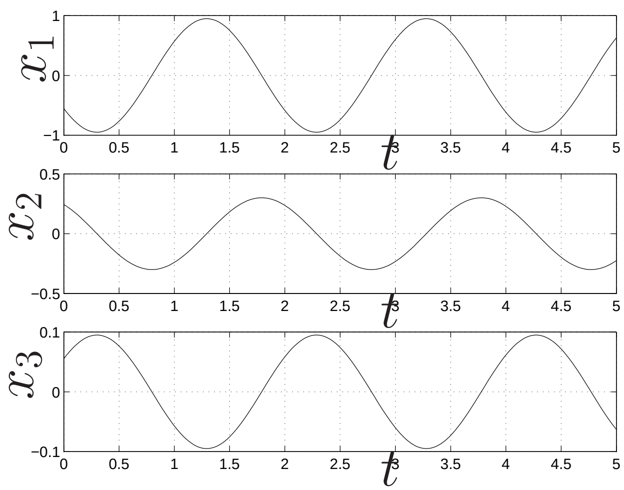
eigenvector associated with eigenvalue $j\sqrt{10}$ is

$$v = \begin{bmatrix} -0.554 + j0.771 \\ 0.244 + j0.175 \\ 0.055 - j0.077 \end{bmatrix}$$

so an invariant plane is spanned by

$$v_{\text{re}} = \begin{bmatrix} -0.554 \\ 0.244 \\ 0.055 \end{bmatrix}, \quad v_{\text{im}} = \begin{bmatrix} 0.771 \\ 0.175 \\ -0.077 \end{bmatrix}$$

for example, with $x(0) = v_{\text{re}}$ we have



Diagonalization Simplifies Matrix Expressions

powers (*i.e.*, discrete-time solution):

$$\begin{aligned} A^k &= \left(T\Lambda T^{-1}\right)^k \\ &= \left(T\Lambda T^{-1}\right)\cdots\left(T\Lambda T^{-1}\right) \\ &= T\Lambda^k T^{-1} \\ &= T\,\mathrm{diag}(\lambda_1^k,\ldots,\lambda_n^k)T^{-1} \end{aligned}$$

(for $k < 0$ only if A invertible, *i.e.*, all $\lambda_i \neq 0$)

exponential (*i.e.*, continuous-time solution):

$$\begin{aligned} e^A &= I + A + A^2/2! + \cdots \\ &= I + T\Lambda T^{-1} + \left(T\Lambda T^{-1}\right)^2/2! + \cdots \\ &= T(I + \Lambda + \Lambda^2/2! + \cdots)T^{-1} \\ &= Te^{\Lambda}T^{-1} \\ &= T\,\mathrm{diag}(e^{\lambda_1},\ldots,e^{\lambda_n})T^{-1} \end{aligned}$$

Stability

we say system $\dot{x} = Ax$ is *stable* if $e^{tA} \to 0$ as $t \to \infty$

meaning:

- state $x(t)$ converges to 0, as $t \to \infty$, no matter what $x(0)$ is
- all trajectories of $\dot{x} = Ax$ converge to 0 as $t \to \infty$

fact: $\dot{x} = Ax$ is stable if and only if all eigenvalues of A have negative real part:

$$\Re\lambda_i < 0, \quad i = 1, \ldots, n$$

Easy to see for diagonalizable matrices; in fact true for all matrices.

Stability of discrete-time systems

suppose A diagonalizable

consider discrete-time LDS $x(t+1) = Ax(t)$

if $A = T\Lambda T^{-1}$, then $A^k = T\Lambda^k T^{-1}$

then

$$x(t) = A^t x(0) = T\Lambda^k T^{-1} x(0) \to 0 \quad \text{as } t \to \infty$$

for all $x(0)$ if and only if

$$|\lambda_i| < 1, \quad i = 1, \ldots, n.$$

again, this is true even when A is not diagonalizable, so we have

fact: $x(t+1) = Ax(t)$ is stable if and only if all eigenvalues of A have magnitude less than one