

11. Matrix Exponential

- Diagonalization
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- Matrix exponential
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- Stability

Diagonalization

suppose v_1, \dots, v_n is a *linearly independent* set of eigenvectors of $A \in \mathbb{R}^{n \times n}$:

$$Av_i = \lambda_i v_i, \quad i = 1, \dots, n$$

express as

$$A \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \cdots & \\ & & \lambda_n \end{bmatrix}$$

define $T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, so

$$AT = T\Lambda$$

- hence $T^{-1}AT = \Lambda$
- T invertible since v_1, \dots, v_n linearly independent
- similarity transformation by T *diagonalizes* A

Modal form

Suppose A is diagonalizable by T . Define new coordinates by $x = T\tilde{x}$, so

$$T\dot{\tilde{x}} = AT\tilde{x} \Leftrightarrow \dot{\tilde{x}} = T^{-1}AT\tilde{x}$$

in new coordinate system, system is diagonal

$$\dot{\tilde{x}} = \Lambda\tilde{x}$$

- Called *modal form*; trajectories consist of n independent modes, *i.e.*,

$$\tilde{x}_i(t) = e^{\lambda_i t} \tilde{x}_i(0)$$

- if initial state $x(0)$ is an eigenvector v , resulting motion is simple — always on the line spanned by v
- for $\lambda \in \mathbb{R}$, $\lambda < 0$, mode contracts or shrinks as $t \uparrow$
- for $\lambda \in \mathbb{R}$, $\lambda > 0$, mode expands or grows as $t \uparrow$

Matrix exponential

For a square matrix M , the **matrix exponential** is defined to be

$$e^M = I + M + \frac{M^2}{2!} + \dots$$

Then the solution of $\dot{x} = Ax$, with $A \in \mathbb{R}^{n \times n}$ is

$$x(t) = e^{tA}x(0)$$

- the series for e^M converges for all M
- one can check that $e^{tA}x(0)$ satisfies the ODE by differentiating
- natural generalization of scalar case

Properties of the Matrix exponential

- matrix exponential is *meant* to look like scalar exponential
- some things you'd guess hold for the matrix exponential (by analogy with the scalar exponential) do in fact hold
- but **many things you'd guess are wrong**

example: you might guess that $e^{A+B} = e^A e^B$, but it's false (in general)

$$\begin{aligned}
 A &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & B &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
 e^A &= \begin{bmatrix} 0.54 & 0.84 \\ -0.84 & 0.54 \end{bmatrix}, & e^B &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
 e^{A+B} &= \begin{bmatrix} 0.16 & 1.40 \\ -0.70 & 0.16 \end{bmatrix} \neq e^A e^B = \begin{bmatrix} 0.54 & 1.38 \\ -0.84 & -0.30 \end{bmatrix}
 \end{aligned}$$

Properties of the Matrix exponential

- $e^{A+B} = e^A e^B$ if $AB = BA$

- e^A is nonsingular, with inverse

$$(e^A)^{-1} = e^{-A}$$

because

$$e^A e^{-A} = e^{A-A} = e^0 = I$$

- $\frac{d}{dt} e^{tA} = A e^{tA} = e^{tA} A$

- Matlab command: `expm(A)`, *not* `exp(A)`

Example: Matrix exponential

let's find e^A , where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

We have

$$e^A = I + A + \frac{A^2}{2!} + \dots = I + A$$

since $A^2 = A^3 = \dots = 0$

Time transfer property

- For $\dot{x} = Ax$ we know

$$x(t) = e^{tA}x(0)$$

interpretation: the matrix e^{tA} propagates initial condition into state at time t

- More generally we have, for *any* t and h ,

$$x(s+t) = e^{tA}x(s)$$

To see this, apply result above to $z(t) = x(t+s)$

interpretation: the matrix e^{tA} propagates state t seconds forward in time

Sampling a continuous-time system

suppose $\dot{x} = Ax$

sample x at times $t_1 \leq t_2 \leq \dots$: define $z(k) = x(t_k)$

then $z(k+1) = e^{(t_{k+1}-t_k)A}z(k)$

for uniform sampling $t_{k+1} - t_k = h$, so

$$z(k+1) = e^{hA}z(k),$$

a discrete-time LDS (called *discretized version* of continuous-time system)

Complex eigenvectors

suppose $Av = \lambda v$, $v \neq 0$, λ is complex

for $a \in \mathbb{C}$, (complex) trajectory $ae^{\lambda t}v$ satisfies $\dot{x} = Ax$

hence so does (real) trajectory

$$\begin{aligned} x(t) &= \Re(ae^{\lambda t}v) \\ &= e^{\sigma t} \begin{bmatrix} v_{\text{re}} & v_{\text{im}} \end{bmatrix} \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix} \begin{bmatrix} \alpha \\ -\beta \end{bmatrix} \end{aligned}$$

where

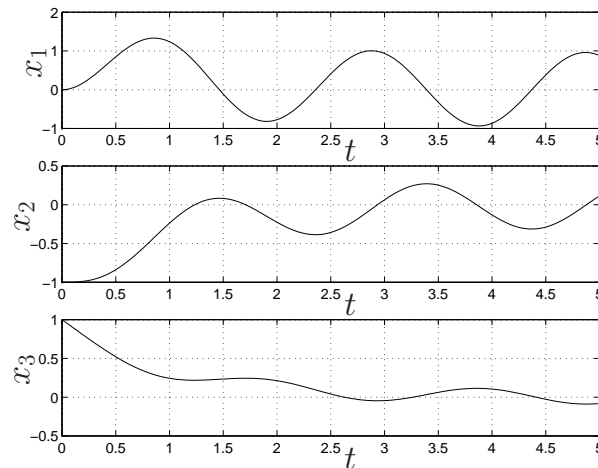
$$v = v_{\text{re}} + jv_{\text{im}}, \quad \lambda = \sigma + j\omega, \quad a = \alpha + j\beta$$

- trajectory stays in *invariant plane* $\text{span}\{v_{\text{re}}, v_{\text{im}}\}$
- σ gives logarithmic growth/decay factor
- ω gives angular velocity of rotation in plane

Example 1

$$\dot{x} = \begin{bmatrix} -1 & -10 & -10 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x \quad \text{eigenvalues are } -1, \pm j\sqrt{10}$$

trajectory with $x(0) = (0, -1, 1)$:

**Example 1**

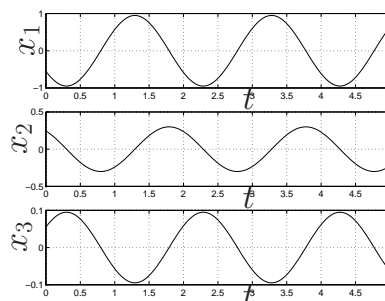
eigenvector associated with eigenvalue $j\sqrt{10}$ is

$$v = \begin{bmatrix} -0.554 + j0.771 \\ 0.244 + j0.175 \\ 0.055 - j0.077 \end{bmatrix}$$

so an invariant plane is spanned by

$$v_{\text{re}} = \begin{bmatrix} -0.554 \\ 0.244 \\ 0.055 \end{bmatrix}, \quad v_{\text{im}} = \begin{bmatrix} 0.771 \\ 0.175 \\ -0.077 \end{bmatrix}$$

for example, with $x(0) = v_{\text{re}}$ we have



Diagonalization Simplifies Matrix Expressions

powers (*i.e.*, discrete-time solution):

$$\begin{aligned} A^k &= (T\Lambda T^{-1})^k \\ &= (T\Lambda T^{-1}) \cdots (T\Lambda T^{-1}) \\ &= T\Lambda^k T^{-1} \\ &= T \operatorname{diag}(\lambda_1^k, \dots, \lambda_n^k) T^{-1} \end{aligned}$$

(for $k < 0$ only if A invertible, *i.e.*, all $\lambda_i \neq 0$)

exponential (*i.e.*, continuous-time solution):

$$\begin{aligned} e^A &= I + A + A^2/2! + \cdots \\ &= I + T\Lambda T^{-1} + (T\Lambda T^{-1})^2/2! + \cdots \\ &= T(I + \Lambda + \Lambda^2/2! + \cdots)T^{-1} \\ &= T e^\Lambda T^{-1} \\ &= T \operatorname{diag}(e^{\lambda_1}, \dots, e^{\lambda_n}) T^{-1} \end{aligned}$$

Stability

we say system $\dot{x} = Ax$ is *stable* if $e^{tA} \rightarrow 0$ as $t \rightarrow \infty$

meaning:

- state $x(t)$ converges to 0, as $t \rightarrow \infty$, no matter what $x(0)$ is
- all trajectories of $\dot{x} = Ax$ converge to 0 as $t \rightarrow \infty$

fact: $\dot{x} = Ax$ is stable if and only if all eigenvalues of A have negative real part:

$$\Re \lambda_i < 0, \quad i = 1, \dots, n$$

Easy to see for diagonalizable matrices; in fact true for all matrices.

Stability of discrete-time systems

suppose A diagonalizable

consider discrete-time LDS $x(t+1) = Ax(t)$

if $A = T\Lambda T^{-1}$, then $A^k = T\Lambda^k T^{-1}$

then

$$x(t) = A^t x(0) = T\Lambda^t T^{-1} x(0) \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

for all $x(0)$ if and only if

$$|\lambda_i| < 1, \quad i = 1, \dots, n.$$

again, this is true even when A is not diagonalizable, so we have

fact: $x(t+1) = Ax(t)$ is stable if and only if all eigenvalues of A have magnitude less than one