Homework 2

Due Thursday 10/8.

1. ***Solving triangular linear equations.***

   Consider the linear equations $y = Rx$, where $R \in \mathbb{R}^{n\times n}$ is upper triangular and invertible. Suggest a simple algorithm to solve for x given R and y. *Hint:* first find x_n; then find x_{n-1} (remembering that now you know x_n); then find x_{n-2} (remembering that now you know x_n and x_{n-1}); etc. **Remark:** the algorithm you will discover is called *back substitution.* It requires order n^2 floating point operations (flops); most methods for solving $y = Ax$ for general $A \in \mathbb{R}^{n\times n}$ require order n^3 flops.

2. ***Some true/false questions.***

   Determine if the following statements are true or false. What we mean by "true" is that the statement is true for all values of the matrices and vectors given. (You can assume the entries of the matrices and vectors are all real.) You can't assume anything about the dimensions of the matrices (unless it's explicitly stated), but you can assume that the dimensions are such that all expressions make sense. For example, the statement "$A + B = B + A$" is true, because no matter what the dimensions of A and B (which must, however, be the same), and no matter what values A and B have, the statement holds. As another example, the statement $A^2 = A$ is false, because there are (square) matrices for which this doesn't hold. (There are also matrices for which it does hold, *e.g.*, an identity matrix. But that doesn't make the statement true.)

   (a) If all coefficients (*i.e.*, entries) of the matrices A and B are nonnegative, and both A and B are onto, then $A + B$ is onto.

   (b) $\operatorname{null}\left(\begin{bmatrix} A \\ A+B \\ A+B+C \end{bmatrix}\right) = \operatorname{null}(A) \cap \operatorname{null}(B) \cap \operatorname{null}(C)$.

   (c) $\operatorname{null}\left(\begin{bmatrix} A \\ AB \\ ABC \end{bmatrix}\right) = \operatorname{null}(A) \cap \operatorname{null}(B) \cap \operatorname{null}(C)$.

   (d) $\operatorname{null}(B^TA^TAB + B^TB) = \operatorname{null}(B)$.

   (e) If $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is full rank, then so are the matrices A and B.

   (f) If $\begin{bmatrix} A & 0 \end{bmatrix}$ is onto, then A is full rank.

   (g) If A^2 is onto, then A is onto.

   (h) If A^TA is onto, then A is onto.

   (i) Suppose $u_1, \ldots, u_k \in \mathbb{R}^n$ are nonzero vectors such that $u_i^Tu_j \geq 0$ for all i, j. Then the vectors are *nonnegative independent*, which means if $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ are nonnegative scalars, and $\sum_{i=1}^k \alpha_i u_i = 0$, then $\alpha_i = 0$ for $i = 1, \ldots, k$.

   (j) Suppose $A \in \mathbb{R}^{n\times k}$ and $B \in \mathbb{R}^{n\times m}$ are skinny, full rank matrices that satisfy $A^TB = 0$. Then $[A\ B]$ is skinny and full rank.

3. ***Right inverses.***

   This problem concerns the specific matrix

$$A = \begin{bmatrix} -1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

   This matrix is full rank (*i.e.*, its rank is 3), so there exists at least one right inverse. In fact, there are many right inverses of A, which opens the possibility that we can seek right inverses that in addition have other properties. For each of the cases below, either find a specific matrix $B \in \mathbb{R}^{5\times 3}$ that satisfies $AB = I$ and the given property, or explain why there is no such B. In cases where there is a right inverse B with the required property, you must briefly explain how you found your B. You must also attach a printout of some Matlab scripts that show the verification that $AB = I$. (We'll be very angry if we have to type in your 5×3 matrix into Matlab to check it.) When there is no right inverse with the given property, briefly explain why there is no such B.

(a) The second row of B is zero.

(b) The nullspace of B has dimension one.

(c) The third column of B is zero.

(d) The second and third rows of B are the same.

(e) B is upper triangular, *i.e.*, $B_{ij} = 0$ for $i > j$.

(f) B is lower triangular, *i.e.*, $B_{ij} = 0$ for $i < j$.

4. ***Single sensor failure detection and identification.***

We have $y = Ax$, where $A \in \mathbb{R}^{m\times n}$ is known, and $x \in \mathbb{R}^n$ is to be found. Unfortunately, up to one sensor may have failed (but you don't know which one has failed, or even whether any has failed). You are given $\tilde{y}$ and not y, where $\tilde{y}$ is the same as y in all entries except, possibly, one (say, the kth entry). If all sensors are operating correctly, we have $y = \tilde{y}$. If the kth sensor fails, we have $\tilde{y}_i = y_i$ for all $i \neq k$.

The file `one_bad_sensor.m`, available on the course web site, defines A and $\tilde{y}$ (as `A` and `ytilde`). Determine which sensor has failed (or if no sensors have failed). You must explain your method, and submit your code.

For this exercise, you can use the Matlab code `rank([F g])==rank(F)` to check if $g \in \operatorname{range}(F)$. (We will see later a much better way to check if $g \in \operatorname{range}(F)$.)

5. ***Householder reflections.***

A *Householder matrix* is defined as

$$Q = I - 2uu^T,$$

where $u \in \mathbb{R}^n$ is normalized, that is, $u^T u = 1$.

(a) Show that Q is orthogonal.

(b) Show that $Qu = -u$. Show that $Qv = v$, for any v such that $u^T v = 0$. Thus, multiplication by Q gives reflection through the plane with normal vector u.

(c) Show that $\det Q = -1$.

(d) Given a vector $x \in \mathbb{R}^n$, find a unit-length vector u for which Qx lies on the line through e_1. *Hint:* Try a u of the form $u = v/\|v\|$, with $v = x + \alpha e_1$ (find the appropriate α and show that such a u works ...) Compute such a u for $x = (3, 2, 4, 1, 5)$. Apply the corresponding Householder reflection to x to find Qx.

Note: Multiplication by an orthogonal matrix has very good numerical properties, in the sense that it does not accumulate much roundoff error. For this reason, Householder reflections are used as building blocks for fast, numerically sound algorithms.

6. ***Channel equalizer with disturbance rejection.***

A communication channel is described by $y = Ax + v$ where $x \in \mathbb{R}^n$ is the (unknown) transmitted signal, $y \in \mathbb{R}^m$ is the (known) received signal, $v \in \mathbb{R}^m$ is the (unknown) disturbance signal, and $A \in \mathbb{R}^{m\times n}$ describes the (known) channel. The disturbance v is known to be a linear combination of some (known) disturbance patterns,

$$d_1, \ldots, d_k \in \mathbb{R}^m.$$

We consider linear equalizers for the channel, which have the form $\hat{x} = By$, where $B \in \mathbb{R}^{n\times m}$. (We'll refer to the matrix B as the equalizer; more precisely, you might say that B_{ij} are the equalizer coefficients.) We say the equalizer B *rejects* the disturbance pattern d_i if $\hat{x} = x$, no matter what x is, when $v = d_i$. If the equalizer rejects a set of disturbance patterns, for example, disturbances d_1, d_3, and d_7 (say), then it can reconstruct the transmitted signal exactly, when the disturbance v is any linear combination of d_1, d_3, and d_7. Here is the problem. For the problem data given in `cedr_data.m`, find an equalizer B that rejects as many disturbance patterns as possible. (The disturbance patterns are given as an $m \times k$ matrix `D`, whose columns are the individual disturbance patterns.) Give the specific set of disturbance patterns that your equalizer rejects, as in 'My equalizer rejects three disturbance patterns: d_2, d_3, and d_6.' (We only need *one* set of disturbances of the maximum size.) Explain how you know that there is no equalizer that rejects more disturbance patterns than yours does. Show the Matlab verification that your B does indeed reconstruct x, and rejects the disturbance patterns you claim it does. Show any other calculations needed to verify that your equalizer rejects the maximum number of patterns possible.