Homework 2 Solutions

Due Thursday 10/8.

1. ***Solving triangular linear equations.***

   Consider the linear equations $y = Rx$, where $R \in \mathbb{R}^{n\times n}$ is upper triangular and invertible. Suggest a simple algorithm to solve for x given R and y. *Hint:* first find x_n; then find x_{n-1} (remembering that now you know x_n); then find x_{n-2} (remembering that now you know x_n and x_{n-1}); etc. **Remark:** the algorithm you will discover is called *back substitution*. It requires order n^2 floating point operations (flops); most methods for solving $y = Ax$ for general $A \in \mathbb{R}^{n\times n}$ require order n^3 flops.

   Solution.

   Suppose that

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & r_{nn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

   Consider the linear equation corresponding to the last (nth) row of R, *i.e.*,

$$y_n = r_{nn}x_n.$$

   If $r_{nn} \neq 0$ we can simply solve for x_n to get $x_n = y_n/r_{nn}$. If $r_{nn} = 0$ then two cases may occur. Either $y_n \neq 0$ which implies that the set of linear equations is *inconsistent* or $y_n = 0$ which implies that the choice of x_n is arbitrary. In any case, $r_{nn} = 0$ means that a unique solution does not exist for the set of linear equations. Now consider the linear equation corresponding to the $(n-1)$th row of R, *i.e.*,

$$y_{n-1} = r_{(n-1)(n-1)}x_{n-1} + r_{(n-1)n}x_n$$

   and for $r_{(n-1)(n-1)} \neq 0$ we get

$$x_{n-1} = \frac{1}{r_{(n-1)(n-1)}}\left(y_{n-1} - r_{(n-1)n}x_n\right)$$

   with x_n found from the previous step. Again if $r_{(n-1)(n-1)} = 0$ it can be said that the system of linear equations has no unique solution. In general, if $x_n, x_{n-1}, \ldots, x_{i+1}$ are known, x_i can be derived from the linear equation corresponding to the ith row of R as (assuming $r_{ii} \neq 0$)

$$x_i = \frac{1}{r_{ii}}\left(y_i - r_{i(i-1)}x_{i-1} - r_{i(i-2)i}x_{i-2} - \cdots - r_{in}x_n\right).$$

   Therefore, the x_i's can be computed recursively for $i = n, n-1, \ldots, 1$ by *back substitution*. This suggests the following simple algorithm:

   $i := n$;
   while $i \geq 1$
       if $r_{ii} \neq 0$
           $x_i := \frac{1}{r_{ii}}\left(y_i - \sum_{j=i+1}^{n} r_{ij}x_j\right)$;
       else
           unique solution does not exist; **break**;
       end
       $i := i - 1$;
   end

Note that whenever $r_{ii} = 0$ a solution does not exist or the solution is not unique. We know that the condition for a (unique) solution to exist is $\det R = \prod_{i=1}^{n} r_{ii} \neq 0$ which also implies that none of the diagonal elements r_{ii} of R to be zero.

2. ***Some true/false questions.***

Determine if the following statements are true or false. What we mean by "true" is that the statement is true for all values of the matrices and vectors given. (You can assume the entries of the matrices and vectors are all real.) You can't assume anything about the dimensions of the matrices (unless it's explicitly stated), but you can assume that the dimensions are such that all expressions make sense. For example, the statement "$A + B = B + A$" is true, because no matter what the dimensions of A and B (which must, however, be the same), and no matter what values A and B have, the statement holds. As another example, the statement $A^2 = A$ is false, because there are (square) matrices for which this doesn't hold. (There are also matrices for which it does hold, *e.g.*, an identity matrix. But that doesn't make the statement true.)

(a) If all coefficients (*i.e.*, entries) of the matrices A and B are nonnegative, and both A and B are onto, then $A + B$ is onto.

(b) $\text{null}\left(\begin{bmatrix} A \\ A+B \\ A+B+C \end{bmatrix}\right) = \text{null}(A) \cap \text{null}(B) \cap \text{null}(C)$.

(c) $\text{null}\left(\begin{bmatrix} A \\ AB \\ ABC \end{bmatrix}\right) = \text{null}(A) \cap \text{null}(B) \cap \text{null}(C)$.

(d) $\text{null}(B^T A^T A B + B^T B) = \text{null}(B)$.

(e) If $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is full rank, then so are the matrices A and B.

(f) If $\begin{bmatrix} A & 0 \end{bmatrix}$ is onto, then A is full rank.

(g) If A^2 is onto, then A is onto.

(h) If $A^T A$ is onto, then A is onto.

(i) Suppose $u_1, \ldots, u_k \in \mathbb{R}^n$ are nonzero vectors such that $u_i^T u_j \geq 0$ for all i, j. Then the vectors are *nonnegative independent*, which means if $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ are nonnegative scalars, and $\sum_{i=1}^{k} \alpha_i u_i = 0$, then $\alpha_i = 0$ for $i = 1, \ldots, k$.

(j) Suppose $A \in \mathbb{R}^{n \times k}$ and $B \in \mathbb{R}^{n \times m}$ are skinny, full rank matrices that satisfy $A^T B = 0$. Then $[A\ B]$ is skinny and full rank.

Solution.

(a) **False.** The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are onto and with nonnegative coefficients but the sum $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has rank one, and so it cannot be onto.

(b) **True.** First let's show that if $x \in \text{null}(A) \cap \text{null}(B) \cap \text{null}(C)$, then $x \in \text{null}\left(\begin{bmatrix} A \\ A+B \\ A+B+C \end{bmatrix}\right)$.

Since $x \in \text{null}(A) \cap \text{null}(B) \cap \text{null}(C)$, we have $Ax = 0$, $Bx = 0$, and $Cx = 0$. Therefore we also have $Ax + Bx = 0 + 0 = 0$, and also $Ax + Bx + Cx = 0 + 0 + 0 = 0$. Therefore we have

$$\begin{bmatrix} A \\ A+B \\ A+B+C \end{bmatrix} x = \begin{bmatrix} Ax \\ Ax+Bx \\ Ax+Bx+Cx \end{bmatrix} = 0.$$

Now let's show the other way around. Suppose $x \in \text{null}\left(\begin{bmatrix} A \\ A+B \\ A+B+C \end{bmatrix}\right)$, then $\begin{bmatrix} A \\ A+B \\ A+B+C \end{bmatrix} x = \begin{bmatrix} Ax \\ Ax+Bx \\ Ax+Bx+Cx \end{bmatrix} = 0$. This means that $Ax = 0$, $Ax+Bx = 0$, and $Ax+Bx+Cx = 0$. Therefore we also have $Ax+Bx = 0+Bx = Bx = 0$, and $Ax+Bx+Cx = 0+0+Cx = Cx = 0$. So $x \in \text{null}(A) \cap \text{null}(B) \cap \text{null}(C)$.

(c) **False.** Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 0 \end{bmatrix}$. The nullspace of A consists of all multiples of $[0\ 1]^T$ and the nullspace of B consists of all multiples of $[1\ 0]^T$. So the intersection of null(A) and null(B) contains only the zero vector. But we also have:

$$\begin{bmatrix} A \\ AB \\ ABC \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and the vector $[0\ 1]^T$ belongs to its nullspace.

(d) **True.** First we show that $\text{null}(A^TA) = \text{null}(A)$. It is clear that if x belongs to the nullspace of A it also belongs to the nullspace of A^TA. So suppose x belongs to the nullspace of A^TA, then:

$$0 = A^TAx \Rightarrow 0 = x^TA^TAx = \|Ax\|_2.$$

This means that $Ax = 0$, which is what we wanted to show. Now we show that:

$$\text{null}\left(\begin{bmatrix} AB \\ B \end{bmatrix}\right) = \text{null}(B).$$

If x is in null(B), then $Bx = 0$, and also $ABx = A0 = 0$. Therefore $\begin{bmatrix} AB \\ B \end{bmatrix} x = \begin{bmatrix} ABx \\ Bx \end{bmatrix} = 0$. Vice versa suppose $x \in \text{null}\left(\begin{bmatrix} AB \\ B \end{bmatrix}\right)$, then

$$\begin{bmatrix} AB \\ B \end{bmatrix} x = \begin{bmatrix} ABx \\ Bx \end{bmatrix} = 0.$$

Therefore $Bx = 0$ and so $x \in \text{null}(B)$. To prove the statement we simply consider

$$\begin{bmatrix} AB \\ B \end{bmatrix}^T \begin{bmatrix} AB \\ B \end{bmatrix} = B^TA^TAB + B^TB,$$

which proves that $\text{null}(B^TA^TAB + B^TB) = \text{null}(B)$.

(e) **True.** If $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ has full rank we know that either the rows or the columns are independent. Suppose the columns are independent (the case for which the rows are independent can be done in the same way considering $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^T$), then the only solution of the equation

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. But we can rewrite the equation as:

$$\begin{aligned} Ax_1 + 0x_2 &= Ax_1 &= 0, \\ 0x_1 + Bx_2 &= Bx_2 &= 0, \end{aligned}$$

therefore the equation $Ax_1 = 0$ has only one solution, $x_1 = 0$, and $Bx_2 = 0$ has only one solution, $x_2 = 0$. This means that the columns of A and the columns of B are independent, and so A and B are full rank.

(f) **True.** If $\begin{bmatrix} A & 0 \end{bmatrix}$ is onto we can solve the equation:

$$y = \begin{bmatrix} A & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax_1 + 0x_2 = Ax_1$$

for all y and so we can also solve $y = Ax_1$, but that means A is onto and so full rank.

(g) **True.** If A^2 is onto, it means that we can solve the equation

$$y = A^2 x$$

for all y, so we can also find a solution for $y = A\hat{x}$ because $y = A^2 x = A(Ax)$, and so we can choose $\hat{x} = Ax$. But this means that A is also onto.

(h) **False.** Take $A = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$. We have $A^T A = 1$ and so $A^T A$ is onto but A is clearly not onto.

(i) **True.** Consider:

$$0 = \sum_{i=1}^{k} \alpha_i u_i \Rightarrow 0 = u_j^T \sum_{i=1}^{k} \alpha_i u_i = \sum_{i \neq j} \alpha_i u_j^T u_i + \alpha_j \|u_j\|^2.$$

Since each term in the summation is nonnegative and $\|u_j\|^2 > 0$, it means that α_j must be zero. Proceeding in the same way for all u_j, we find $\alpha_j = 0$ for $j = 1, \ldots, k$.

(j) **True.** To say that A is skinny and full rank means that its k columns are independent, and span a k-dimensional subspace of $\mathbb{R}^n$, range(A). We can also say that the columns of B are independent, and span a m-dimensional subspace of $\mathbb{R}^n$, range(B). To say that $A^T B = 0$ means that every column of A is orthogonal to every column of B, which means that range$(A) \perp$ range(B). It follows that the span of all the columns of A and B put together, *i.e.*, range$([A\ B])$, has dimension $k + m$. This means that $k+m \leq n$ and also that $[A\ B]$ is full rank. Here's another proof. Assume the hypotheses hold. To say that $[A\ B]$ is not skinny and full rank means that its columns are dependent. That means there exists a nonzero vector $[v^T\ w^T]^T$ such that

$$[A\ B] \begin{bmatrix} v \\ w \end{bmatrix} = Av + Bw = 0.$$

Thus we have $Av = -Bw$. Multiplying on the left by A^T we get $A^T Av = -A^T Bw = 0$ since $A^T B = 0$. Since A is skinny and full rank, $A^T A$ is nonsingular, and' we conclude that $v = 0$, which in turn means $w = 0$. But this contradicts our assumption that $[v^T\ w^T]^T$ is nonzero.

3. ***Right inverses.***

This problem concerns the specific matrix

$$A = \begin{bmatrix} -1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

This matrix is full rank (*i.e.*, its rank is 3), so there exists at least one right inverse. In fact, there are many right inverses of A, which opens the possibility that we can seek right inverses that in addition have other properties. For each of the cases below, either find a specific matrix $B \in \mathbb{R}^{5\times 3}$ that satisfies $AB = I$ and the given property, or explain why there is no such B. In cases where there is a right inverse B with the required property, you must briefly explain how you found your B. You must also attach a printout of some Matlab scripts that show the verification that $AB = I$. (We'll be very angry if we have to type in your 5×3 matrix into Matlab to check it.) When there is no right inverse with the given property, briefly explain why there is no such B.

(a) The second row of B is zero.

(b) The nullspace of B has dimension one.

(c) The third column of B is zero.

(d) The second and third rows of B are the same.

(e) B is upper triangular, *i.e.*, $B_{ij} = 0$ for $i > j$.

(f) B is lower triangular, *i.e.*, $B_{ij} = 0$ for $i < j$.

Solution.

(a) The second row of B is zero. This means that the second column of A isn't used in forming AB. Let $\tilde{A}$ be the matrix A with its second column removed, and let $\tilde{B}$ denote the matrix B with its second row (which is supposed to be zero) removed. We have $\tilde{A}\tilde{B} = AB = I$, so $\tilde{B}$ is a right inverse of $\tilde{A}$. There is such a matrix if and only if $\tilde{A}$ is full rank, which it is. We can take $\tilde{B} = \tilde{A}^T(\tilde{A}\tilde{A}^T)^{-1}$. Finally to construct B we simply insert a zero second row, moving rows 2, 3, 4 down by one. This gives the matrix

$$B = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}.$$

There are other possible choices as well.

(b) The nullspace of B has dimension one. This means that B has rank 2, so the rank of AB is at most 2, which rules out the possibility that $AB = I$. So this is impossible.

(c) The third column of B is zero. This implies B has a nullspace with dimension at least one, so by part (b) above, this is impossible too.

(d) The second and third rows of B are the same. Let $\tilde{B}$ denote B with one of the (identical) rows 2 and 3 deleted. Then we have $AB = \tilde{A}\tilde{B}$, where $\tilde{A}$ is obtained from the matrix A by replacing its second column with the sum of its second and third columns, and deleting its third column. Thus, we need to find a right inverse for $\tilde{A}$, provided it is full rank. It is, so we can take $\tilde{B} = \tilde{A}^T(\tilde{A}\tilde{A}^T)^{-1}$. Finally to construct B we simply insert a second copy of the second row of $\tilde{B}$ as a new third row. This gives

$$B = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}.$$

This matrix also happens to be the pseudo-inverse of A, $B = A^T(AA^T)^{-1}$, and some of you noticed this immediately and used the pseudo-inverse to answer this question. That's a fine answer; it was our mistake to choose A so that the pseudo-inverse satisfied this condition. In general, of course, it would not.

(e) B is upper triangular, *i.e.*, $B_{ij} = 0$ for $i > j$. If B is upper triangular, then it has the form

$$\begin{bmatrix} \tilde{B} \\ 0 \end{bmatrix},$$

where $\tilde{B}$ is square and upper triangular. If $AB = I$, then $\tilde{A}\tilde{B} = I$, where $\tilde{A}$ is the matrix formed from the first 3 columns of A. Thus we have $\tilde{A} = \tilde{B}^{-1}$. But the inverse of an upper triangular matrix is also upper triangular, so unless $\tilde{A}$ is upper triangular (and it isn't, in this case), we can't possibly have $\tilde{A}\tilde{B} = I$. So there is no such B in this case.

(f) B is lower triangular, *i.e.*, $B_{ij} = 0$ for $i < j$. Let's label the columns of B as

$$b_1, \qquad b_2 = \begin{bmatrix} 0 \\ \tilde{b}_2 \end{bmatrix}, \qquad b_3 = \begin{bmatrix} 0 \\ 0 \\ \tilde{b}_3 \end{bmatrix},$$

where $\tilde{b}_2 \in \mathbb{R}^4$ and $\tilde{b}_3 \in \mathbb{R}^3$. To say that $AB = I$ is the same as saying that $Ab_1 = e_1$, $Ab_2 = e_2$, and $Ab_3 = e_3$, where e_1, e_2, e_3 are the unit vectors. We can solve these equations separately. The first equation is easy; the second we reduce to $\tilde{A}\tilde{b}_2 = e_2$, where here $\tilde{A}$ is A with its first column removed. The third is handled similarly. These equations do have a solution; we get

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Another way: we set it up as a set of 9 linear equations (one for each entry of $AB = I$) in $5 + 4 + 3 = 12$ variables. The variables are the first column of B (with 5 entries), the nonzero part of the second column of B (with 4 entries), and the nonzero part of the third second column of B (with 3 entries). We then attempt to solve these 9 equations in 12 variables. Some equations immediately give us the B matrix coefficients, while the others can be solved by inspection to obtain a rather simple matrix

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

4. ***Single sensor failure detection and identification.***

We have $y = Ax$, where $A \in \mathbb{R}^{m \times n}$ is known, and $x \in \mathbb{R}^n$ is to be found. Unfortunately, up to one sensor may have failed (but you don't know which one has failed, or even whether any has failed). You are given $\tilde{y}$ and not y, where $\tilde{y}$ is the same as y in all entries except, possibly, one (say, the kth entry). If all sensors are operating correctly, we have $y = \tilde{y}$. If the kth sensor fails, we have $\tilde{y}_i = y_i$ for all $i \neq k$.

The file `one_bad_sensor.m`, available on the course web site, defines A and $\tilde{y}$ (as `A` and `ytilde`). Determine which sensor has failed (or if no sensors have failed). You must explain your method, and submit your code.

For this exercise, you can use the Matlab code `rank([F g])==rank(F)` to check if $g \in \text{range}(F)$. (We will see later a much better way to check if $g \in \text{range}(F)$.)

Solution.

Let $y^{(i)}$ be the measurement vector y with the ith entry removed. Likewise, let $A^{(i)}$ be the measurement matrix with the ith row of A removed. This corresponds to the system without the ith sensor.

If the ith sensor is faulty, we will almost surely have $y \notin \text{range}(A)$ (unless the sensor failure happens to give the same response y_i as that predicted by A, which is highly unlikely). However, once we remove its faulty measurement, we will certainly have $y^{(i)} \in \text{range}(A^{(i)})$.

To test if a vector z is in range(C), we can use Matlab and compare `rank([C z]) == rank(C)`. If they are equal, $z \in \text{range}(C)$. Otherwise `rank([C z]) == rank(C) + 1`. To find a faulty sensor, we remove one row of A at a time, and use the above test.

The 11th sensor is faulty.

5. ***Householder reflections.***

A *Householder matrix* is defined as

$$Q = I - 2uu^T,$$

where $u \in \mathbb{R}^n$ is normalized, that is, $u^T u = 1$.

(a) Show that Q is orthogonal.

(b) Show that $Qu = -u$. Show that $Qv = v$, for any v such that $u^T v = 0$. Thus, multiplication by Q gives reflection through the plane with normal vector u.

(c) Show that $\det Q = -1$.

(d) Given a vector $x \in \mathbb{R}^n$, find a unit-length vector u for which Qx lies on the line through e_1. *Hint:* Try a u of the form $u = v/\|v\|$, with $v = x + \alpha e_1$ (find the appropriate α and show that such a u works ...) Compute such a u for $x = (3, 2, 4, 1, 5)$. Apply the corresponding Householder reflection to x to find Qx.

Note: Multiplication by an orthogonal matrix has very good numerical properties, in the sense that it does not accumulate much roundoff error. For this reason, Householder reflections are used as building blocks for fast, numerically sound algorithms.

Solution.

(a)

$$\begin{aligned}
Q^TQ &= (I - 2uu^T)^T(I - 2uu^T) \\
&= (I - 2uu^T)(I - 2uu^T) \\
&= I - 2uu^T - 2uu^T + 4uu^Tuu^T \\
&= I - 2uu^T - 2uu^T + 4uu^T \qquad \text{using } u^Tu = 1 \\
&= I \qquad \text{so Q is orthogonal}
\end{aligned}$$

(b)

$$\begin{aligned}
Qu &= u - 2uu^Tu = u - 2u = -u \qquad \text{using } u^Tu = 1 \\
Qv &= v - 2uu^tv = v \qquad \text{using } u^Tv = 0
\end{aligned}$$

(c) We know $\det(Q) = \prod_{i=1}^n \lambda_i$. Since Q is symmetric, all eigenvalues are real and we can construct an orthonormal eigenvector basis. From parts (a) and (b), u is an eigenvector with associated eigenvalue -1, and any vector v orthogonal to u is an eigenvector with associated eigenvalue 1. The nullspace of u^T has dimension $n-1$, so we can construct an orthogonal eigenbasis with all eigenvalues 1 except for the -1 eigenvalue with eigenvector u. Thus the product of the eigenvalues is $-1 = \det(Q)$.

(d) Since Q is orthogonal, $Q^TQ = I$ has all eigenvalues 1, hence all singular values of Q are 1, so $\kappa(Q) = 1$ (*i.e.*, Q is as well-conditioned as can be.)

(e) We follow the hint and choose $u = (x + \alpha e_1)/\|x + \alpha e_1\|$. Then

$$\begin{aligned}
Q &= I - 2\frac{(x+\alpha e_1)(x+\alpha e_1)^T}{(x+\alpha e_1)^T(x+\alpha e_1)} \\
&= I - 2\frac{x(x^T + \alpha e_1^T) + \alpha e_1(x^T + \alpha e_1^T)}{x^Tx + \alpha e_1^Tx + \alpha x^Te_1 + \alpha^2 e_1^Te_1} \\
Qx &= x - 2\frac{x(\|x\|^2 + \alpha e_1^Tx) + e_1(\alpha\|x\|^2 + \alpha^2 x_1}{\|x\|^2 + 2\alpha x_1 + \alpha^2} \\
&= x - \frac{2\|x\|^2 + 2\alpha x_1}{\|x\|^2 + 2\alpha x_1 + \alpha^2}x - 2\alpha\frac{\|x\|^2 + \alpha x_1}{\|x\|^2 + 2\alpha x_1 + \alpha^2}e_1 \\
&= \underbrace{\left(1 - \frac{2\|x\|^2 + 2\alpha x_1}{\|x\|^2 + 2\alpha x_1 + \alpha^2}\right)}_{\text{Need this zero}} x - 2\alpha\frac{\|x\|^2 + \alpha x_1}{\|x\|^2 + 2\alpha x_1 + \alpha^2}e_1
\end{aligned}$$

We can achieve this by choosing $\alpha = \pm\|x\|$. This leads to $Qx = \mp\|x\|e_1$ (which makes sense …Q should always preserve norm). Some people used a geometric argument here as well, and this can make the solution a lot neater if it's well presented. The idea is to find a reflection plane that reflects the given vector onto the e_1 axis (there are two possibilities, for negative and positive parts of the e_1 axis), and u is then a unit vector orthogonal to this plane.

6. ***Channel equalizer with disturbance rejection.***

A communication channel is described by $y = Ax + v$ where $x \in \mathbb{R}^n$ is the (unknown) transmitted signal, $y \in \mathbb{R}^m$ is the (known) received signal, $v \in \mathbb{R}^m$ is the (unknown) disturbance signal, and $A \in \mathbb{R}^{m\times n}$ describes the (known) channel. The disturbance v is known to be a linear combination of some (known) disturbance patterns,

$$d_1, \dots, d_k \in \mathbb{R}^m.$$

We consider linear equalizers for the channel, which have the form $\hat{x} = By$, where $B \in \mathbb{R}^{n\times m}$. (We'll refer to the matrix B as the equalizer; more precisely, you might say that B_{ij} are the equalizer coefficients.) We say the equalizer B *rejects* the disturbance pattern d_i if $\hat{x} = x$, no matter what x is, when $v = d_i$. If the equalizer rejects a set of disturbance patterns, for example, disturbances d_1, d_3, and d_7 (say), then it can reconstruct the transmitted signal exactly, when the disturbance v is any linear combination of d_1, d_3, and d_7. Here is the problem. For the problem data given in `cedr_data.m`, find an equalizer B that rejects as many disturbance patterns as possible. (The disturbance patterns are given as an $m \times k$ matrix `D`, whose columns are the individual disturbance patterns.) Give the specific set of disturbance patterns that your equalizer rejects, as in 'My equalizer rejects three disturbance patterns: d_2, d_3, and d_6.' (We only need *one* set of disturbances of the maximum size.) Explain how you know that there is no equalizer that rejects more disturbance patterns than yours does. Show the Matlab verification that your B does indeed reconstruct x, and rejects the disturbance patterns you claim it does. Show any other calculations needed to verify that your equalizer rejects the maximum number of patterns possible.

Solution.

Solution. The condition that A reconstructs x exactly, with no disturbances is $BA = I$. This implies that BA is full rank, and so are A and B. The equalizer rejects disturbance d_k only if $Bd_k = 0$. Thus, the equalizer rejects disturbances $d_{k_1}, \dots, d_{k_r}$ provided

$$B[A\ \tilde{D}] = [I\ 0], \qquad (1)$$

where

$$\tilde{D} = \begin{bmatrix} d_{k_1} & \cdots & d_{k_r} \end{bmatrix}.$$

So the problem comes down to finding a matrix B that satisfies (1) for the largest value of r. If we have a set of p independent disturbances that are rejected each of them has to belong to the nullspace of B and therefore the dimension of the nullspace of B has to be larger than or equal to p. Moreover since B is full rank we have that the size of the nullspace of B is $m - n$. Therefore we cannot have more than $m - n$ independent disturbances rejected by B. In summary: if the disturbances are independent, then we cannot reject more than $m - n$ of them. This makes good intuitive sense: our original signal has n degrees of freedom, and we get $m > n$ measurements. This leaves us, roughly speaking, $m - n$ degrees of freedom. These can be used to reject at most $m - n$ disturbances. The next question is, can we reject $m-n$ disturbances, assuming A has rank n? If we can, then that's our solution, because above we argued that you cannot reject more than $m - n$ disturbances. This is the case if and only if there is a set of $r = m - n$ disturbances that is independent of the columns of A. Let's show this. Suppose now that we have $m - n$ disturbances and we construct

$$[A\ \tilde{D}] \in \mathbb{R}^{n\times n}.$$

If $[A\ \tilde{D}]$ is nonsingular, *i.e.*, the columns $d_{k_1}, \ldots, d_{k_r}$ are independent, and also independent of the columns of A, we can invert it. Let's define B and E from its inverse as

$$[A\ \tilde{D}]^{-1} = \begin{bmatrix} B \\ E \end{bmatrix}.$$

We can now check that B satisfies equation (1) and is therefore a valid decoder. In fact, since

$$I = [A\ \tilde{D}]^{-1}[A\ \tilde{D}] = \begin{bmatrix} B \\ E \end{bmatrix}[A\ \tilde{D}],$$

we have

$$B[A\ \tilde{D}] = [I\ 0].$$

Conversely, if $[A\ \tilde{D}]$ is singular we cannot solve for

$$B[A\ \tilde{D}] = [I\ 0],$$

because the matrix $[I\ 0]$ is full rank. We now address the numerical problem provided. The first thing we need to do is to check if the set of disturbances is independent. If this is the case we have a bound on the maximum number of patterns that can be rejected. The following code checks if the columns of D are independent.

```
clear all
close all
chan_equal_rej_data
n
m
k
rankd=rank(D)
```

The output of this code is

```
n =
7
m =
12
k =
10
rankd =
10
```

This means that the maximum number of disturbances that can be rejected in this case cannot be greater than $m - n = 5$. The following code checks if it's possible to rejects the first 5 disturbances in the matrix D. We define $C = [A\ \tilde{D}]$.

```
C=[A D(:,1:5)];
rank(C)
```

Since that output of the code is

```
ans =
12
```

C is full rank and we can construct an equalizer B. The following Matlab code solves for B and check is B is an equalizer.

```
Cinv=inv(C);
B=Cinv(1:n,:);
norm(B*A-eye(n))
norm(B*D(:,1:5)-zeros(n,5))
```

The output of the code is

```
ans =
4.1181e-15
ans =
7.2002e-15
```

which shows that the matrix B is an equalizer for the matrix A, that rejects the first 5 disturbances $d_1, \ldots, d_5$. In the numerical problem provided, it turns out that with *any* set of 5 disturbances, the matrix $[A\ D]$ is invertible. In other words: for this problem, we can reject *any* set of 5 disturbances. But we can't reject any set of 6 disturbances. We need to mention one very common mistake. Many of you got $BA = I$, and then asserted or assumed that $B = (A^T A)^{-1} A^T$, *i.e.*, the pseudo-inverse of A. Of course, that matrix is *a* left inverse of A, but it is not the only one. And for this problem, it's not a good one at all, since it rejects *none* of the given disturbances. We call this the 'Let's use least-squares and hope for the best' approach. In this problem, it doesn't work.