Homework 3

Due Thursday 10/15.

1. ***Some basic properties of eigenvalues.***

   Show that

   (a) the eigenvalues of A and A^T are the same

   (b) A is invertible if and only if A does not have a zero eigenvalue

   (c) if the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$ and A is invertible, then the eigenvalues of A^{-1} are $1/\lambda_1, \ldots, 1/\lambda_n$,

   (d) the eigenvalues of A and $T^{-1}AT$ are the same.

   Hint: you'll need to use the facts that $\det A = \det(A^T)$, $\det(AB) = \det A \det B$, and, if A is invertible, $\det A^{-1} = 1/\det A$.

2. ***Tridiagonal Toeplitz matrices***

   Some matrices have simple formulae for eigenvalues and eigenvectors. An example we have seen is the circulant matrices. Another example is given by tridiagonal Toeplitz matrices, as follows. Suppose

$$G = \begin{bmatrix} b & a & & & & \\ c & b & a & & & \\ & c & b & a & & \\ & & & \ddots & & \\ & & & & & a \\ & & & & c & b \end{bmatrix}$$

   is an $n \times n$ real matrix, and $a \neq 0$ and $c \neq 0$.

   (a) Suppose

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

   is an eigenvector of G. Show that

$$cx_{k-1} + bx_k + ax_{k+1} = \lambda x_k$$

   for all $k = 1, \ldots, n$, where we define for convenience $x_0 = 0$ and $x_{n+1} = 0$. Hence show that the solution x must be of the form

$$x_k = \alpha r_1^k + \beta r_2^k$$

   for some choice of α, β, r_1 and r^2.

   (b) Hence show that r_1 and r_2 satisfy

$$r_1 = \sqrt{c/a}e^{i\pi m/(n+1)} \qquad r_2 = \sqrt{c/a}e^{-i\pi m/(n+1)}$$

   for some m with $1 \leq m \leq n$. Here $i = \sqrt{-1}$.

   (c) Hence show that the eigenvalues λ_k of G are

$$\lambda_k = b + 2a\sqrt{c/a}\cos\left(\frac{k\pi}{n+1}\right)$$

   for $k = 1 \ldots, n$.

3. ***Norm expressions for quadratic forms.***

   Let $f(x) = x^T Ax$ (with $A = A^T \in \mathbb{R}^{n\times n}$) be a quadratic form.

   (a) Show that f is positive semidefinite (*i.e.*, $A \geq 0$) if and only if it can be expressed as $f(x) = \|Fx\|^2$ for some matrix $F \in \mathbb{R}^{k\times n}$. Explain how to find such an F (when $A \geq 0$). What is the size of the smallest such F (*i.e.*, how small can k be)?

   (b) Show that f can be expressed as a difference of squared norms, in the form $f(x) = \|Fx\|^2 - \|Gx\|^2$, for some appropriate matrices F and G. How small can the sizes of F and G be?

4. ***Positive semidefinite (PSD) matrices.***

   (a) Show that if A and B are PSD and $\alpha \in \mathbb{R}$, $\alpha \geq 0$, then so are αA and $A + B$.

   (b) Show that any (symmetric) submatrix of a PSD matrix is PSD. (To form a symmetric submatrix, choose any subset of $\{1, \dots, n\}$ and then throw away all other columns and rows.)

   (c) Show that if $A \geq 0$, $A_{ii} \geq 0$.

   (d) Show that if $A \geq 0$, $|A_{ij}| \leq \sqrt{A_{ii}A_{jj}}$. In particular, if $A_{ii} = 0$, then the entire ith row and column of A are zero.

5. ***Drawing ellipsoids in Matlab.***

   Write a MATLAB function which, given a 2×2 real symmetric matrix B, plots the ellipse

   $$E = \left\{ x \in \mathbb{R}^2 \mid x^T B^{-1} x = 1 \right\}$$

   along with the major and minor axes. Your code should be short (maybe five lines). Test it using the following script

```
B=[4 1 ; 1 2];
figure(1);
clf;
hold on;
myellipse(B);
grid;
axis equal;
```

   Hint: It's easiest to write code to draw a circle first. If B is symmetric, what is the image of the unit ball under B? Is that the ellipsoid you need?

6. ***Optimal time compression equalizer.***

   We are given the (finite) impulse response of a communications channel, *i.e.*, the real numbers

   $$c_1, c_2, \dots, c_n.$$

   Our goal is to design the (finite) impulse response of an equalizer, *i.e.*, the real numbers

   $$w_1, w_2, \dots, w_n.$$

   (To make things simple, the equalizer has the same length as the channel.) The equalized channel response h is given by the convolution of w and c, *i.e.*,

   $$h_i = \sum_{j=1}^{i-1} w_j c_{i-j}, \quad i = 2, \dots, 2n,$$

   where we take w_i and c_i to be zero for $i \leq 0$ or $i > n$. This is shown below.

The goal is to choose w so that most of the energy of the equalized impulse response h is *concentrated* within k samples of $t = n + 1$, where $k < n - 1$ is given. To define this formally, we first define the total energy of the equalized response as

$$E_{\mathrm{tot}} = \sum_{i=2}^{2n} h_i^2,$$

and the energy in the desired time interval as

$$E_{\mathrm{des}} = \sum_{i=n+1-k}^{n+1+k} h_i^2.$$

For any w for which $E_{\mathrm{tot}} > 0$, we define the *desired to total energy ratio*, or DTE, as $\mathrm{DTE} = E_{\mathrm{des}}/E_{\mathrm{tot}}$. Thus number is clearly between 0 and 1; it tells us what fraction of the energy in h is contained in the time interval $t = n+1-k, \ldots, t = n+1+k$. You can assume that h is such that for any $w \neq 0$, we have $E_{\mathrm{tot}} > 0$.

(a) How do you find a $w \neq 0$ that maximizes DTE? You must give a very clear description of your method, and explain why it works. Your description and justification must be *very clear*. You can appeal to any concepts used in the class, *e.g.*, least-squares, least-norm, eigenvalues and eigenvectors, singular values and singular vectors, matrix exponential, and so on.

(b) Carry out your method for time compression length $k = 1$ on the data found in `time_comp_data.m`. Plot your solution w, the equalized response h, and give the DTE for your w.

Please note: You do not need to know anything about equalizers, communications channels, or even convolution; everything you need to solve this problem is clearly defined in the problem statement.