Homework 3 Solutions

Due Thursday 10/15.

1. ***Some basic properties of eigenvalues.***

   Show that

   (a) the eigenvalues of A and A^T are the same

   (b) A is invertible if and only if A does not have a zero eigenvalue

   (c) if the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$ and A is invertible, then the eigenvalues of A^{-1} are $1/\lambda_1, \ldots, 1/\lambda_n$,

   (d) the eigenvalues of A and $T^{-1}AT$ are the same.

   Hint: you'll need to use the facts that $\det A = \det(A^T)$, $\det(AB) = \det A \det B$, and, if A is invertible, $\det A^{-1} = 1/\det A$.

   Solution.

   (a) The eigenvalues of a matrix A are given by the roots of the polynomial $\det(sI - A)$. From determinant properties we know that $\det(sI - A) = \det(sI - A)^T = \det(sI - A^T)$. We conclude that the eigenvalues of A and A^T are the same.

   (b) First we recall that A is invertible if and only if $\det(A) \neq 0$. But $\det(A) \neq 0 \iff \det(-A) \neq 0$.

      i. If 0 is an eigenvalue of A, then $\det(sI - A) = 0$ when $s = 0$. It follows that $\det(-A) = 0$ and thus $\det(A) = 0$, and A is not invertible. From this fact we conclude that if A is invertible, then 0 is not an eigenvalue of A.

      ii. If A is not invertible, then $\det(A) = \det(-A) = 0$. This means that, for $s = 0$, $\det(sI - A) = 0$, and we conclude that in this case 0 must be an eigenvalue of A. From this fact it follows that if 0 is not an eigenvalue of A, then A is invertible.

   (c) From the results of the last item we see that 0 is not an eigenvalue of A. Now consider the eigenvalue/eigenvector pair (λ_i, x_i) of A. This pair satisfies $Ax_i = \lambda_i x_i$. Now, since A is invertible, λ_i is invertible. Multiplying both sides by A^{-1} and λ_i^{-1} we have $\lambda_i^{-1} x_i = A^{-1} x_i$, and from this we conclude that the eigenvalues of the inverse are the inverse of the eigenvalues.

   (d) First we note that $\det(sI - A) = \det(I(sI - A)) = \det(T^{-1}T(sI - A))$. Now, from determinant properties, we have $\det(T^{-1}T(sI - A)) = \det(T^{-1}(sI - A)T)$. But this is also equal to $\det(sI - T^{-1}AT)$, and the conclusion is that the eigenvalues of A and $T^{-1}AT$ are the same.

2. ***Tridiagonal Toeplitz matrices***

   Some matrices have simple formulae for eigenvalues and eigenvectors. An example we have seen is the circulant matrices. Another example is given by tridiagonal Toeplitz matrices, as follows. Suppose

$$G = \begin{bmatrix} b & a & & & & \\ c & b & a & & & \\ & c & b & a & & \\ & & & \ddots & & \\ & & & & & a \\ & & & & c & b \end{bmatrix}$$

   is an $n \times n$ real matrix, and $a \neq 0$ and $c \neq 0$.

(a) Suppose

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is an eigenvector of G. Show that

$$cx_{k-1} + bx_k + ax_{k+1} = \lambda x_k$$

for all $k = 1, \ldots, n$, where we define for convenience $x_0 = 0$ and $x_{n+1} = 0$. Hence show that the solution x must be of the form

$$x_k = \alpha r_1^k + \beta r_2^k$$

for some choice of α, β, r_1 and r^2.

(b) Hence show that r_1 and r_2 satisfy

$$r_1 = \sqrt{c/a}e^{i\pi m/(n+1)} \qquad r_2 = \sqrt{c/a}e^{-i\pi m/(n+1)}$$

for some m with $1 \leq m \leq n$. Here $i = \sqrt{-1}$.

(c) Hence show that the eigenvalues λ_k of G are

$$\lambda_k = b + 2a\sqrt{c/a}\cos\left(\frac{k\pi}{n+1}\right)$$

for $k = 1 \ldots, n$.

Solution.

(a) We can solve the difference equation in the same manner that we would a differential equation of the same form. We guess a solution of

$$x_k = r^k$$

Plugging this into the difference equation yields

$$r^{k-1}(c + (b - \lambda)r + ar^2) = 0$$

There are three cases for the roots of the above quadratic; two distinct real roots, one repeated root, or two complex conjugate roots. In the case of a repeated root, every solution x_k has the form

$$x_k = \alpha r^k + \beta k r^k$$

However, when the boundary conditions of the problem are considered

$$x_0 = 0 \qquad x_{n+1} = 0$$

the coefficients α and β must equal zero, and hence $x = 0$. Since eigenvectors are nonzero, this is not the solution we need. The same argument shows that the above quadratic cannot have distinct real roots. Hence, the solution of this problem has r_1 and r_2 being complex conjugates, and

$$x_k = \alpha r_1^k + \beta r_2^k$$

as desired.

(b) The solution of the quadratic is

$$r_1 = \gamma + \sqrt{\gamma^2 - \frac{c}{a}} \qquad r_2 = \gamma - \sqrt{\gamma^2 - \frac{c}{a}}$$

where

$$\gamma = \frac{\lambda - b}{2a}$$

For convenience, write these in polar coordinates as

$$r_1 = \sqrt{\frac{c}{a}} e^{i\phi} \qquad r_2 = \sqrt{\frac{c}{a}} e^{-i\phi}$$

Since we know that $r_1 r_2 = c/a$ we have

$$\cos(\phi) = \frac{\gamma}{\sqrt{\frac{c}{a}}}$$

The boundary conditions imply that

$$\alpha + \beta = 0 \qquad r_1^{n+1} - r_2^{n+1} = 0$$

and therefore

$$\left(\frac{r_1}{r_2}\right)^{n+1} = 1$$

Hence,

$$\phi = \frac{\pi m}{n+1}$$

for some m with $1 \le m \le n$.

(c) From the above solution for ϕ, we have

$$\cos\left(\frac{\pi m}{n+1}\right) = \frac{\lambda - b}{2a\sqrt{\frac{c}{a}}}$$

Hence, the eigenvalues λ_m of G are

$$\lambda_m = b + 2a\sqrt{\frac{c}{a}} \cos\left(\frac{\pi m}{n+1}\right)$$

for $m = 1, \dots, n$.

3. ***Norm expressions for quadratic forms.***

Let $f(x) = x^T A x$ (with $A = A^T \in \mathbb{R}^{n \times n}$) be a quadratic form.

(a) Show that f is positive semidefinite (*i.e.*, $A \ge 0$) if and only if it can be expressed as $f(x) = \|Fx\|^2$ for some matrix $F \in \mathbb{R}^{k \times n}$. Explain how to find such an F (when $A \ge 0$). What is the size of the smallest such F (*i.e.*, how small can k be)?

(b) Show that f can be expressed as a difference of squared norms, in the form $f(x) = \|Fx\|^2 - \|Gx\|^2$, for some appropriate matrices F and G. How small can the sizes of F and G be?

Solution.

(a) We know that the norm expression $f(x) = \|Fx\|^2$ is a positive semidefinite quadratic form simply because $f(x) \ge 0$ for all x and $f(x) = x^T A x$ with $A = F^T F \ge 0$. In this problem we will show the converse, *i.e.*, any positive semidefinite quadratic form $f(x) = x^T A x$ can be written as a norm expression $f(x) = \|Fx\|^2$. Suppose the eigenvalue decomposition of $A \ge 0$ is $Q \Lambda Q^T$, with $Q^T Q = I$ and $\Lambda = \mathbf{diag}(\lambda_1, \dots, \lambda_n)$

where λ_i are the eigenvalues of A. Since $\lambda_i \geq 0$ (because $A \geq 0$) then $\Lambda^{1/2} = \mathbf{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$ is a real matrix. Let $F = \Lambda^{1/2} Q^T \in \mathbb{R}^{n \times n}$. Then we have $\|Fx\|^2 = x^T F^T F x = Q\Lambda^{1/2}\Lambda^{1/2}Q^T = x^T A x = f(x)$. To get smallest F suppose that $\mathrm{rank}(A) = r$. Therefore, $A \in \mathbb{R}^{n\times n}$ has exactly r nonzero eigenvalues $\lambda_1, \ldots, \lambda_r$. Suppose $\Lambda_+ = \mathbf{diag}(\lambda_1, \ldots, \lambda_r)$. Hence, the eigenvalue decomposition of A can be written as

$$A = \left[\begin{array}{c|c} Q_1 & Q_2 \end{array}\right] \left[\begin{array}{c|c} \Lambda_+ & 0_{r\times(n-r)} \\ \hline 0_{(n-r)\times r} & 0_{(n-r)\times(n-r)} \end{array}\right] \left[\begin{array}{c} Q_1^T \\ \hline Q_2^T \end{array}\right]$$

and as a result $A = Q_1 \Lambda_+ Q_1^T$ where $Q_1 \in \mathbb{R}^{n\times r}$. Now we can take $F = \Lambda_+^{1/2} Q_1^T \in \mathbb{R}^{r\times n}$. Therefore, k can be as small as r, *i.e.*, $k = \mathrm{rank}(r)$. Note that k cannot be any smaller than $\mathrm{rank}(A)$ because $A = F^T F$ implies that $\mathrm{rank}(A) \leq k$.

(b) In general, a quadratic form need not to be positive semidefinite. In this problem we show that any quadratic form can be decomposed into its "positive" and "negative" parts. In other words, we can write $f(x)$ as the difference of two norm expressions, *i.e.*, $f(x) = \|Fx\|^2 - \|Gx\|^2$. Suppose A has n_1 positive eigenvalues $\lambda_1, \ldots, \lambda_{n_1}$, n_2 negative eigenvalues $\lambda_{n_1+1}, \ldots, \lambda_{n_1+n_2}$, and therefore $n - n_1 - n_2$ zero eigenvalues. Let

$$\Lambda_+ = \mathbf{diag}(\lambda_1, \ldots, \lambda_{n_1}), \quad \Lambda_- = \mathbf{diag}(-\lambda_{n_1+1}, \ldots, -\lambda_{n_1+n_2}).$$

The eigenvalue decomposition of A can be written as

$$A = \left[\begin{array}{c|c|c} Q_1 & Q_2 & Q_3 \end{array}\right] \left[\begin{array}{c|c|c} \Lambda_+ & 0_{n_1\times n_2} & 0_{n_1\times(n-n_1-n_2)} \\ \hline 0_{n_2\times n_1} & -\Lambda_- & 0_{n_2\times(n-n_1-n_2)} \\ \hline 0_{(n-n_1-n_2)\times n_1} & 0_{(n-n_1-n_2)\times n_2} & 0_{(n-n_1-n_2)\times(n-n_1-n_2)} \end{array}\right] \left[\begin{array}{c} Q_1^T \\ \hline Q_2^T \\ \hline Q_3^T \end{array}\right]$$

so $A = Q_1 \Lambda_+ Q_1^T - Q_2^T \Lambda_- Q_2$. Now simply take $F = \Lambda_+^{1/2} Q_1^T \in \mathbb{R}^{n_1\times n}$ and $G = \Lambda_-^{1/2} Q_2^T \in \mathbb{R}^{n_2\times n}$. It is easy to verify that $A = F^T F - G^T G$ and therefore $x^T A x = \|Fx\|^2 - \|Gx\|^2$. In fact, this method gives the smallest sizes for F and G.

4. ***Positive semidefinite (PSD) matrices.***

(a) Show that if A and B are PSD and $\alpha \in \mathbb{R}$, $\alpha \geq 0$, then so are αA and $A + B$.

(b) Show that any (symmetric) submatrix of a PSD matrix is PSD. (To form a symmetric submatrix, choose any subset of $\{1, \ldots, n\}$ and then throw away all other columns and rows.)

(c) Show that if $A \geq 0$, $A_{ii} \geq 0$.

(d) Show that if $A \geq 0$, $|A_{ij}| \leq \sqrt{A_{ii}A_{jj}}$. In particular, if $A_{ii} = 0$, then the entire ith row and column of A are zero.

Solution.

(a) To show that $\alpha A \geq 0$ we verify that $x^T(\alpha A)x \geq 0$ for all x. But $x^T(\alpha A)x = \alpha(x^T A x)$ and since $x^T A x \geq 0$ ($A \geq 0$) and $\alpha \geq 0$, we immediately get $x^T(\alpha A)x \geq 0$. Again, to show that $A + B \geq 0$ we show that $x^T(A + B)x \geq 0$ for all x. This is easy because $x^T(A + B)x = x^T A x + x^T B x$ and $A, B \geq 0$ imply that $x^T A x, x^T B x \geq 0$ and therefore $x^T(A + B)x \geq 0$.

(b) Suppose that $A = A^T \geq 0$. Any symmetric submatrix of A can be written as $Z^T A Z$ for some suitable matrix Z. For example, if $A \in \mathbb{R}^{3\times 3}$ and we want to pick the submatrix formed by the first and third columns and rows we simply take

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

so that

$$Z^TAZ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{13} \\ A_{13} & A_{33} \end{bmatrix}.$$

The idea here is to pick the columns of Z as the unit vectors corresponding to the column/row numbers we want to keep. In this example, we wanted to keep the first and third columns/rows so we took $Z = [e_1\ e_3]$. In general, consider the $m \times m$ symmetric submatrix of A which consists of elements of A that are only on the columns and rows $i_1, \ldots, i_m$ of A. Then it is easy to verify that

$$\text{(submatrix formed from columns/rows } i_1, \ldots, i_m) = Z^TAZ, \quad Z = [e_{i_1} \ \cdots \ e_{i_m}],$$

where e_{i_j} is the i_jth unit vector in $\mathbb{R}^n$. Using the result of problem (**??**), $A \geq 0$ implies that $Z^TAZ \geq 0$ and therefore any symmetric submatrix of A is also positive semidefinite.

(c) This is easy. We can simply use the result of the previous part ($A_{ii} \in \mathbb{R}$ is a 1×1 symmetric submatrix of A), or more directly, use the fact that $A \geq 0$ implies $e_i^TAe_i \geq 0$ and note that $e_i^TAe_i$ is nothing but A_{ii}.

(d) Choose any 2×2 symmetric submatrix of A, say

$$\tilde{A} = \left[\begin{bmatrix} A_{ii} & A_{ij} \\ A_{ij} & A_{jj} \end{bmatrix}\right].$$

According to problem (**??**) this (symmetric) submatrix is positive semidefinite and therefore its eigenvalues are nonnegative. Hence, the determinant of the submatrix (which is equal to the product of the eigenvalues) is also nonnegative. In other words

$$\det \tilde{A} = \left[\begin{bmatrix} A_{ii} & A_{ij} \\ A_{ij} & A_{jj} \end{bmatrix}\right] = A_{ii}A_{jj} - A_{ij}^2 \geq 0$$

and immediately we get $A_{ij}^2 \leq A_{ii}A_{jj}$ or $|A_{ij}| \leq \sqrt{A_{ii}A_{jj}}$. In particular, if $A_{ii} = 0$ then $|A_{ij}| \leq 0$ or $A_{ij} = 0$ (for any j) and the entire ith row (and hence ith column since A is symmetric) should be zero.

5. ***Drawing ellipsoids in Matlab.***

Write a MATLAB function which, given a 2×2 real symmetric matrix B, plots the ellipse

$$E = \left\{ x \in \mathbb{R}^2 \mid x^TB^{-1}x = 1 \right\}$$

along with the major and minor axes. Your code should be short (maybe five lines). Test it using the following script

```
B=[4 1 ; 1 2];
figure(1);
clf;
hold on;
myellipse(B);
grid;
axis equal;
```

Hint: It's easiest to write code to draw a circle first. If B is symmetric, what is the image of the unit ball under B? Is that the ellipsoid you need?

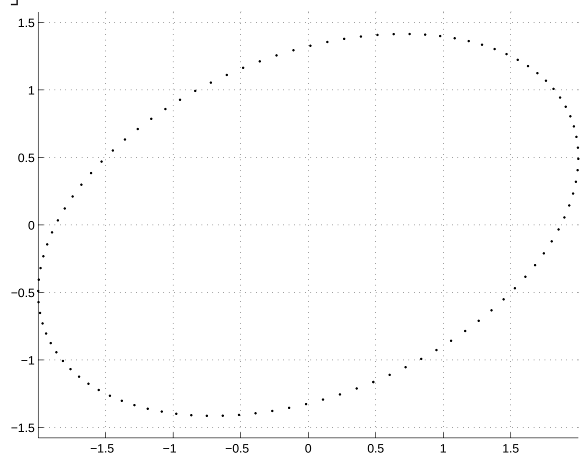
Solution.

One possible Matlab routine is

```
function myellipse(B)
theta=2*pi*[0:0.01:1];
x=[cos(theta); sin(theta)];
[V,D]=eig(B);
x=V*sqrt(D)*x;
plot(x(1,:),x(2,:),'r');

% now plot the axes
ellipse_axes=V*diag(sqrt(diag(D)));
line([0,ellipse_axes(1,1)],[0,ellipse_axes(2,1)]);
line([0,ellipse_axes(1,2)],[0,ellipse_axes(2,2)]);
```

With $B = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$ we have output



6. ***Optimal time compression equalizer.***

We are given the (finite) impulse response of a communications channel, *i.e.*, the real numbers

$$c_1, c_2, \ldots, c_n.$$

Our goal is to design the (finite) impulse response of an equalizer, *i.e.*, the real numbers

$$w_1, w_2, \ldots, w_n.$$

(To make things simple, the equalizer has the same length as the channel.) The equalized channel response h is given by the convolution of w and c, *i.e.*,

$$h_i = \sum_{j=1}^{i-1} w_j c_{i-j}, \quad i = 2, \ldots, 2n,$$

where we take w_i and c_i to be zero for $i \le 0$ or $i > n$. This is shown below.

The goal is to choose w so that most of the energy of the equalized impulse response h is *concentrated* within k samples of $t = n + 1$, where $k < n - 1$ is given. To define this formally, we first define the total energy of the equalized response as

$$E_{\text{tot}} = \sum_{i=2}^{2n} h_i^2,$$

and the energy in the desired time interval as

$$E_{\text{des}} = \sum_{i=n+1-k}^{n+1+k} h_i^2.$$

For any w for which $E_{\text{tot}} > 0$, we define the *desired to total energy ratio*, or DTE, as $\text{DTE} = E_{\text{des}}/E_{\text{tot}}$. Thus number is clearly between 0 and 1; it tells us what fraction of the energy in h is contained in the time interval $t = n+1-k, \dots, t = n+1+k$. You can assume that h is such that for any $w \neq 0$, we have $E_{\text{tot}} > 0$.

(a) How do you find a $w \neq 0$ that maximizes DTE? You must give a very clear description of your method, and explain why it works. Your description and justification must be *very clear*. You can appeal to any concepts used in the class, *e.g.*, least-squares, least-norm, eigenvalues and eigenvectors, singular values and singular vectors, matrix exponential, and so on.

(b) Carry out your method for time compression length $k = 1$ on the data found in `time_comp_data.m`. Plot your solution w, the equalized response h, and give the DTE for your w.

Please note: You do not need to know anything about equalizers, communications channels, or even convolution; everything you need to solve this problem is clearly defined in the problem statement.

Solution.

(a) First we are going to find expressions for E_{tot} and E_{des}. We can write $h = Aw$, where A is the Toeplitz matrix

$$A = \begin{bmatrix} c_1 & & & \\ c_2 & c_1 & & \\ & & \ddots & \\ c_n & c_{n-1} & \cdots & c_1 \\ & c_n & \cdots & c_2 \\ & & \ddots & \\ & & & c_n \end{bmatrix}.$$

Let's define the subvector of the equalized response corresonding to the desired time interval as

$$\bar{h} = \begin{bmatrix} h_{n+1-k} \\ \vdots \\ h_{n+1+k} \end{bmatrix}.$$

Then we have $\bar{h} = Bw$, where B is the Toeplitz matrix consisting of the $n-k, \dots, n+k$ rows of A. Now we have

$$E_{\text{tot}} = \|h\|^2 = h^T h = w^T A^T A w, \qquad E_{\text{des}} = \|\bar{h}\|^2 = \bar{h}^T \bar{h} = w^T B^T B w.$$

The assumption that $E_{\text{tot}} > 0$ for any $w \neq 0$ means that $A^T A > 0$. Now we can state the problem in simple matrix form. We need to find w that maximizes the ratio of quadratic forms

$$d = \frac{w^T B^T B w}{w^T A^T A w}.$$

That's close to, but not exactly, a problem we have studied. We know how to solve this problem in the case $A^T A = I$. So let's reduce our problem to this one. Let's define $z = (A^T A)^{1/2} w$, so we have $w = (A^T A)^{-1/2} z$. We can express the DTE d in terms of z as

$$d = \frac{w^T B^T B w}{w^T A^T A w} = \frac{z^T (A^T A)^{-1/2} B^T B (A^T A)^{-1/2} z}{z^T z}.$$

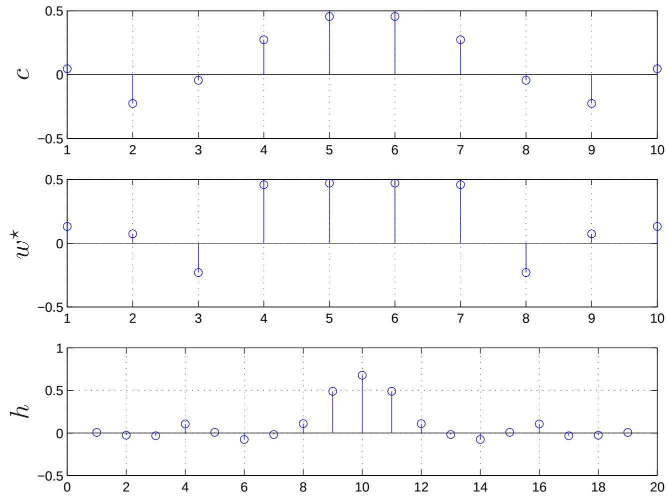


Figure 1: Time compression equilizer.

Now we do know how to maximize this ratio. Its maximum value is

$$d_{\max} = \lambda_{\max}\left((A^TA)^{-1/2}B^TB(A^TA)^{-1/2}\right),$$

and the value of z that maximizes the ratio is v, the eigenvector associated with the maximum eigenvalue above. To find the value of w that maximizes DTE, we multiply v by $(A^TA)^{-1/2}$. Here's the summary:

- Form the symmetric matrix $C = (A^TA)^{-1/2}B^TB(A^TA)^{-1/2}$.
- Let v be the eigenvector of C associated with its largest eigenvalue $\lambda_{\max}$.
- Let $w^\star = (A^TA)^{-1/2}z$.

The same algorithm can also be expressed in terms of the SVD of A and B. Many students came up with heuristics for (approximately) solving this problem, ranging from iterative least-squares, regularization, etc. Some of these methods even came up with an answer close to the correct answer. But we still took off a number of points, since the goal is to describe a correct method, not just to get the particular numerical answer (in this case). One heuristic is to simply take w as the right singular vector of B associated with its largest signular value. This does well (at least for this example) but isn't correct.

(b) When we carry out the procedure described above on the given problem instance with time-compression parameter $k = 1$, we obtain the optimal DTE $d_{\max} = 0.9375$. Thus, 93.75% of the energy in the equalized impulse response h is concentrated within the window of 3 samples around the impulse center. This can also be seen in figure **??**, where we plotted the channel impulse response c, the equalizer $w^\star$, and the equalized impulse response h (also see the Matlab code below).

```
time_comp_data;  % defines channel impulse response
n = length(c);
k = 1; % time compression window length
A = toeplitz([c;zeros(n-1,1)], [c(1);zeros(n-1,1)]);
B = A(n-k:n+k,:);
D = inv(sqrtm(A'*A))*B'*B*inv(sqrtm(A'*A));
[vmax dmax] = eigs(D,1);
wmax = inv(sqrtm(A'*A))*vmax;
figure; subplot(311), stem(c,'o'); ylabel('c'); xlabel('n'); grid;
subplot(312), stem(wmax,'o'); ylabel('w'); xlabel('n'); grid;
```

```
subplot(313), stem(A*wmax,'o'); ylabel('h'); xlabel('n'); grid;
print -depsc time_comp_eq.eps
```