Homework 4

Due Thursday 10/22.

1. ***A Pythagorean inequality for the matrix norm.***

   Suppose that $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times n}$. Show that

   $$\left\| \begin{bmatrix} A \\ B \end{bmatrix} \right\| \le \sqrt{\|A\|^2 + \|B\|^2}.$$

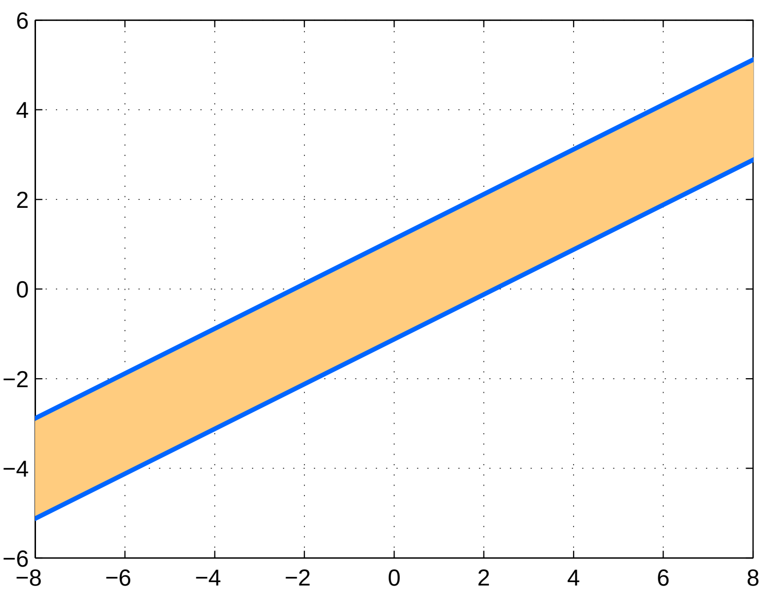
   Under what conditions do we have equality?

2. ***Eigenvalues and singular values of a symmetric matrix.***

   Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues, and let $\sigma_1, \dots, \sigma_n$ be the singular values of a matrix $A \in \mathbb{R}^{n\times n}$, which satisfies $A = A^T$. (The singular values are based on the full SVD: If $\text{rank}(A) < n$, then some of the singular values are zero.) You can assume the eigenvalues (and of course singular values) are sorted, *i.e.*, $\lambda_1 \ge \cdots \ge \lambda_n$ and $\sigma_1 \ge \cdots \ge \sigma_n$. How are the eigenvalues and singular values related?

3. ***Degenerate ellipsoids***

   The picture below shows a degenerate ellipsoid.



   In two dimensions, a degenerate ellipsoid is a *slab*; the sides are parallel lines. For the above example the slab has half-width 1 (i.e., it has width 2) and the center axis points in the direction

   $$v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

   We'll call the slab S.

   (a) Find a symmetric matrix $Q \in \mathbb{R}^{2\times 2}$ such that the slab above is

   $$S = \left\{ x \in \mathbb{R}^2 \mid x^T Q x \le 1 \right\}$$

   (b) Is Q positive definite?

(c) Consider the matrix

$$P = \begin{bmatrix} 0.04 & 0.06 \\ 0.06 & 0.09 \end{bmatrix}$$

Plot the slab corresponding to P. (Sketch it by hand, if you prefer.) What is the axis of the slab? (i.e., the long axis). What is the half-width of the slab?

(d) Now let's consider the three dimensional case. Suppose T is a cylinder, of radius 5, whose axis of symmetry passes through the origin and points along the vector

$$q = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Find a symmetric matrix A such that the cylinder is

$$T = \left\{ x \in \mathbb{R}^3 \mid x^T A x \leq 1 \right\}$$

(e) Just for fun (i.e., not worth any points.) Is A unique? If so prove it; if not, find two different matrices A which both generate the cylinder T above.

4. ***Sensitivity of the force to changes in the position.***

Suppose masses m_1, m_2, m_3, m_4 are located at positions x_1, x_2, x_3, x_4 in a line and connected by springs with spring constants k_{12}, k_{23}, k_{34} whose natural lengths of extension are l_{12}, l_{23}, l_{34}. Let f_1, f_2, f_3, f_4 denote the rightward forces on the m asses, e.g., $f_1 = k_{12}(x_2 - x_1 - l_{12})$.

Write the 4×4 matrix relating the column vectors f and x. Let K denote the matrix in this equation, so that

$$f = Kx + c$$

and we can measure the forces f.

Suppose the vector of positions x changes from x to $x + \delta x$, with a corresponding change in the measured force from f to $f + \delta f$. Let $v_1, \ldots, v_n$ be the right singular vectors of K. Show that if $\delta x \in \operatorname{span}\{v_{k+1}, \ldots, v_n\}$ then $\|\delta f\| \leq \sigma_{k+1} \|\delta x\|$.

Suppose the spring constants $k_{ij} = 1$. What displacement of the position vector δx causes the least change to the forces? Can you interpret this?

5. ***Computing the SVD by hand.***

Consider the matrix

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}$$

(a) Determine, on paper, an SVD of A in the usual form $A = U \Sigma V^T$. The SVD is not unique, so find the one that has the minimal number of minus signs in U and V.

(b) List the singular values, left singular vectors, and right singular vectors of A. Draw a careful, labeled picture of the unit ball in $\mathbb{R}^2$ and its image under A, together with the singular vectors, with the coordinates of their vertices marked.

(c) What is the matrix norm $\|A\|$ and the Frobenius norm $\|A\|_F$?

(d) Find A^{-1}, not directly, but via the SVD.

(e) Find the eigenvalues, λ_1 and λ_2 of A.

(f) Verify that $\det A = \lambda_1 \lambda_2$ and $|\det A| = \sigma_1 \sigma_2$

(g) What is the area of the ellipsoid onto which A maps the unit ball of $\mathbb{R}^2$?

6. ***Using the SVD for numerical computations.***

Use the SVD and Matlab to solve the following problems. Turn in your code.

(a) Find bases for the range space and null space of each of the following matrices:

$$A_1 = \begin{bmatrix} 4 & 1 & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) For the linear equation

$$\begin{bmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

is there a solution? If so find it, and say if it is unique.

(c) Find all solutions of

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

7. ***Determining the number of signal sources.***

The signal transmitted by n sources is measured at m receivers. The signal transmitted by each of the sources at sampling period k, for $k = 1, \ldots, p$, is denoted by an n-vector $x(k) \in \mathbb{R}^n$. The gain from the j-th source to the i-th receiver is denoted by $a_{ij} \in \mathbb{R}$. The signal measured at the receivers is then

$$y(k) = A\,x(k) + v(k), \quad k = 1, \ldots, p,$$

where $v(k) \in \mathbb{R}^m$ is a vector of sensor noises, and $A \in \mathbb{R}^{m \times n}$ is the matrix of source to receiver gains. However, we do not know the gains a_{ij}, nor the transmitted signal $x(k)$, nor even the number of sources present n. We only have the following additional *a priori* information:

- We expect the number of sources to be less than the number of receivers (*i.e.*, $n < m$, so that A is skinny);
- A is full-rank and well-conditioned;
- All sources have roughly the same average power, the signal $x(k)$ is unpredictable, and the source signals are unrelated to each other; Hence, given enough samples (*i.e.*, p large) the vectors $x(k)$ will 'point in all directions';
- The sensor noise $v(k)$ is small relative to the received signal $A\,x(k)$.

Here's the question:

(a) You are given a large number of vectors of sensor measurements $y(k) \in \mathbb{R}^m$, $k = 1, \ldots, p$. How would you estimate the number of sources, n? Be sure to clearly describe your proposed method for determining n, and to explain when and why it works.

(b) Try your method on the signals given in the file `nsources.m`. Running this script will define the variables:

- `m`, the number of receivers;
- `p`, the number of signal samples;
- `Y`, the receiver sensor measurements, an array of size `m` by `p` (the k-th column of `Y` is $y(k)$.)

What can you say about the number of signal sources present?

Note: Our problem description and assumptions are not precise. An important part of this problem is to explain your method, and clarify the assumptions.