3. Linear Algebra Review

- Range and Null space
- Left and right invertibility
- Rank
- Conservation of dimension
- Invertibility

The Range

The range is defined to be

$$\mathrm{range}(A) = \{\, Ax \mid x \in \mathbb{R}^n \,\}$$

for *control* problems:

$$\begin{aligned} \mathrm{range}(A) &= \text{set of possible outputs of } y = Ax \\ &= \text{span of columns of } A \end{aligned}$$

the range is also called the *column space* or the *image* of A

the range is important in control problems because

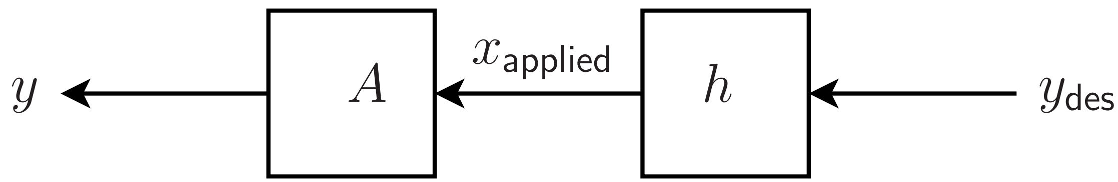
The equation $y = Ax$ has a solution x $\iff$ $y \in \mathrm{range}(A)$

If we want a solution for all $y \in \mathbb{R}^m$, then we need $\mathrm{range}(A) = \mathbb{R}^m$.
We call such a matrix *full-range* or *fully controllable* or *onto* or *surjective* or *right-invertible*

Full Range Space

$\operatorname{range} A = \mathbb{R}^m$ is immediately equivalent to

- the columns of A span $\mathbb{R}^m$.
- there is a *right inverse function* $h : \mathbb{R}^n \to \mathbb{R}^m$ so that $Ah(y) = y$



Notice that h is a *controller* that gives an input that exactly produces y_{des}, since

$$y = Ax_{\text{applied}} = Ah(y_{\text{des}}) = y_{\text{des}}$$

We'll see that in fact there is a *linear* right inverse B such that $AB = I$.

Not so immediately, we'll see that $\operatorname{range} A = \mathbb{R}^m$ if and only if

- the rows of A are linearly independent.
- AA^T is invertible.

The Null Space

$$\operatorname{null}(A) = \{\, x \in \mathbb{R}^n \mid Ax = 0 \,\}$$

for *estimation* problems:

$$\begin{aligned}\operatorname{null}(A) &= \text{set of unknowns which produce zero sensor output} \\ &= \text{set of vectors orthogonal to } \textit{all} \text{ rows of } A\end{aligned}$$

the null space is also called the *kernel* of A.

if $z \in \operatorname{null} A$ then x and $x + z$ are indistinguishable from sensor reading y

the null space is important in estimation problems because

> if x_0 is one solution to $y = Ax$, then the set of all solutions is
> $$\{\, x_0 + z \mid z \in \operatorname{null}(A) \,\}$$

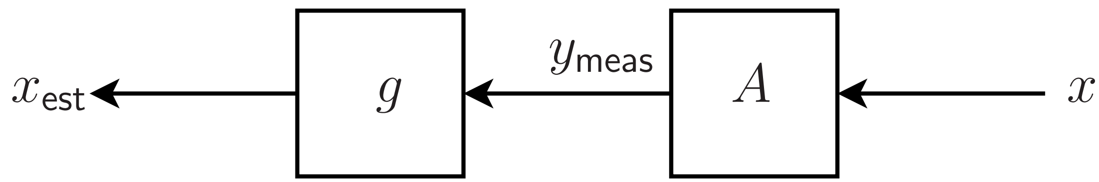
This follows immediately from the definition of null space, and means that we can *unambiguously* determine x from y if and only if $\operatorname{null}(A) = \{0\}$.

We call A *zero-nullspace* or *fully observable* or *one-to-one* or *injective* or *left-invertible*

Zero Null Space

$\mathrm{null}(A) = \{0\}$ is immediately equivalent to

- the columns of A are linearly independent.
- there is a *left-inverse function*, a function $g : \mathbb{R}^m \to \mathbb{R}^n$ so that $g(Ax) = x$



Notice that the function g is an estimator that can exactly reconstruct x, because

$$x_{\text{est}} = g(y_{\text{meas}}) = g(Ax) = x$$

We'll see that there is a *linear* left inverse C such that $CA = I$.

We'll see that $\mathrm{null}\, A = \{0\}$ if and only if

- the rows of A span $\mathbb{R}^n$, that is, $\mathrm{range}(A^T) = \mathbb{R}^n$.
- $A^T A$ is invertible

Subspaces

A subset $S \subset \mathbb{R}^n$ is called a *subspace* if

$$\boxed{\begin{array}{c} x + y \in S \\ \lambda x \in S \end{array}}$$

for all $x, y \in S$ and all $\lambda \in R$.

- A subspace is closed under addition and scalar multiplication
- If $A \in \mathbb{R}^{m \times n}$ then
  - $\operatorname{range} A$ is a subspace of $\mathbb{R}^m$
  - $\operatorname{null} A$ is a subspace of $\mathbb{R}^n$

  These are *numerical representations* of a subspace
- Easy to test if $x \in S$ with these representations...

  How to measure distance of x from S?

Rank

the *rank* of a matrix is the dimension of its column space. i.e.,

$$\operatorname{rank}(A) = \dim \operatorname{range}(A)$$

Terminology

- an $m \times n$ matrix A is called *full rank* if

$$\operatorname{rank}(A) = \min\{m, n\}$$

- A is called *skinny* if $n < m$ and *fat* if $n > m$
- A is called *full column rank* if its columns are linearly independent

Properties:

- Later: the dim. of the column space equals the dimension of the row space. i.e.,

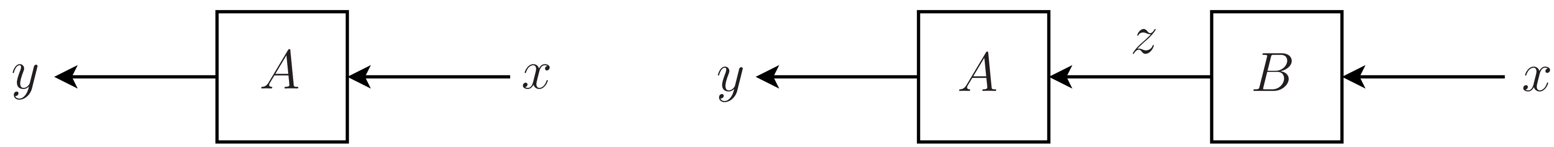
$$\operatorname{rank}(A) = \operatorname{rank}(A^T)$$

Coding interpretation of rank

- easy to see that the rank of product satisfies $\operatorname{rank} BC \leq \min\{\operatorname{rank} B, \operatorname{rank} C\}$

- hence if $A = BC$ with $B \in \mathbb{R}^{m \times r}$, $C \in \mathbb{R}^{r \times n}$, then $\operatorname{rank} A \leq r$

- conversely, we'll see later that if $\operatorname{rank} A = r$ then $A \in \mathbb{R}^{m \times n}$ can be factored as $A = BC$ with $B \in \mathbb{R}^{m \times r}$, $C \in \mathbb{R}^{r \times n}$



- **rank**$(A) = r$ is minimum size of vector needed to faithfully reconstruct y from x

Application: fast matrix-vector multiplication

- need to compute matrix-vector product $y = Ax$, $A \in \mathbb{R}^{m \times n}$
- A has known factorization $A = BC$, $B \in \mathbb{R}^{m \times r}$
- computing $y = Ax$ directly: mn operations
- computing $y = Ax$ as $y = B(Cx)$ (compute $z = Cx$ first, then $y = Bz$): $rn + mr = (m + n)r$ operations
- savings can be considerable if $r \ll \min\{m, n\}$

Conservation of dimension

an important property of rank is

$$\boxed{\dim \operatorname{range}(A) + \dim \operatorname{null}(A) = n}$$

We interpret this as *conservation of dimension*. Of n input dimensions, every one is either mapped to zero or mapped to the output.

Therefore

- if $\operatorname{range} A = \mathbb{R}^m$ then $n \geq m$.
- if $\operatorname{null} A = \{0\}$ then $n \leq m$.
- and hence if both hold, $n = m$.

Proof of conservation of dimension

Suppose $A = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}$ where $a_i \in \mathbb{R}^m$. Then

$$\dim \operatorname{range} A + \dim \operatorname{null} A = n$$

Proof: Let V_1 be a matrix whose columns are a basis for $\operatorname{null} A$, and let V_2 be a matrix whose columns complete that basis to one for $\mathbb{R}^n$. Let

$$V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$

so that $\operatorname{range} V = \mathbb{R}^n$, then

$$\begin{aligned} \operatorname{range} A &= \operatorname{range} AV \\ &= \operatorname{range} \begin{bmatrix} 0 & AV_2 \end{bmatrix} \\ &= \operatorname{range} AV_2 \end{aligned}$$

since $AV_1 = 0$. Further, the columns of AV_2 are linearly independent. To see this, suppose not, that is that there exists a nonzero vector z such that $AV_2 z = 0$. Then we would have $V_2 z \in \operatorname{null} A$, which is impossible. Hence the columns of AV_2 are a basis for $\operatorname{range} A$, and there are $n - \dim \operatorname{null} A$ of them.

Invertibility

If $\operatorname{range} A = \mathbb{R}^m$ and $\operatorname{null} A = \{0\}$ then

- A is square, that is, $m = n$
- A is has a left-inverse $g(Ax) = x$ and a right-inverse $Ah(y) = y$

In fact

- The inverses are equal, $h = g$
- and linear, so there is a matrix B such that

$$AB = BA = I$$

- and B is the unique such matrix

Such a matrix is called *invertible*

Invertibility

Suppose A is left-invertible and right-invertible. Then there is a matrix B such that

$$AB = BA = I$$

Proof: Since $g(Ax) = x$ and $Ah(y) = y$, we have

$$g \circ A \circ h = h \qquad g \circ A \circ g = g$$

Hence $g = h$. Call this function f. To see that f is linear, suppose $Ax_1 = y_1$ and $Ax_2 = y_2$. Then we know

$$A(\alpha x_1 + \beta x_2) = \alpha y_1 + \beta y_2$$

and so

$$\alpha x_1 + \beta x_2 = f(\alpha y_1 + \beta y_2)$$

and so

$$\alpha f(y_1) + \beta f(y_2) = f(\alpha y_1 + \beta y_2)$$

which means f is linear.