

9. Multiobjective Least-Squares

- Multi-objective least squares
- Trade-off curves
- Weighted-sum objectives
- Example: trade-off curve

Multi-Objective Least Squares

in many problems we have two (or more) objectives

- we want $J_1 = \|Ax - y\|^2$ small
- and also $J_2 = \|Fx - g\|^2$ small

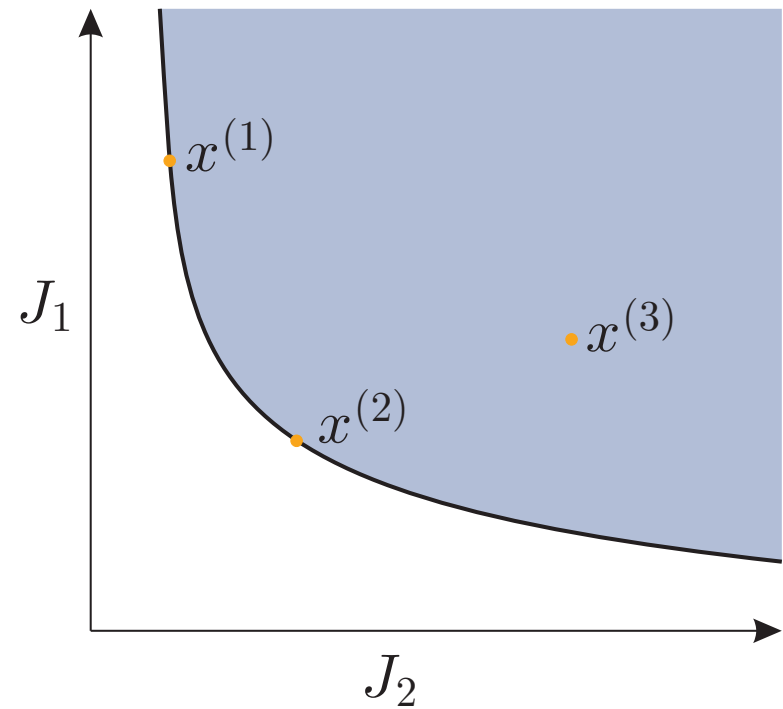
- usually the objectives are *competing*
- we can make one smaller at the expense of making the other larger

common example: $F = I, g = 0$:

we want $\|Ax - y\|$ small, with small x

trade-off curve

- shaded area shows (J_2, J_1) achieved by some $x \in \mathbb{R}^n$
- clear area shows (J_2, J_1) not achieved by any $x \in \mathbb{R}^n$
- boundary of region is called *optimal trade-off curve*
- corresponding x called *Pareto optimal* (for the objectives $\|Ax - y\|^2$, $\|Fx - g\|^2$)



three example choices of x : $x^{(1)}$, $x^{(2)}$, $x^{(3)}$

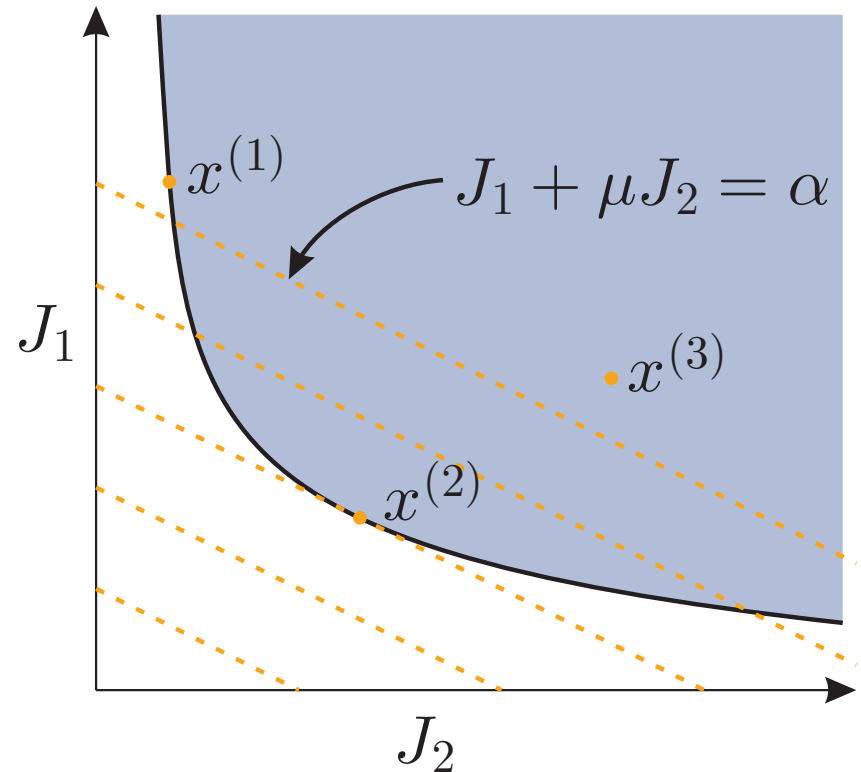
- $x^{(3)}$ is worse than $x^{(2)}$ on both counts (J_2 and J_1)
- $x^{(1)}$ is better than $x^{(2)}$ in J_2 , but worse in J_1

weighted-sum objective

to find Pareto optimal points, i.e. x 's on optimal trade-off curve, we minimize the *weighted-sum* objective:

$$J_1 + \mu J_2 = \|Ax - y\|^2 + \mu \|Fx - g\|^2$$

parameter $\mu \geq 0$ gives relative weight between J_1 and J_2



points where weighted sum is constant, $J_1 + \mu J_2 = \alpha$ correspond to line with slope $-\mu$

- $x^{(2)}$ minimizes the weighted-sum objective for μ shown
- by varying μ from 0 to $+\infty$, we can sweep out the entire *optimal trade-off curve*

minimizing weighted-sum objective

can express weighted-sum objective as ordinary least-squares objective

$$\begin{aligned}\|Ax - y\|^2 + \mu\|Fx - g\|^2 &= \left\| \begin{bmatrix} A \\ \sqrt{\mu}F \end{bmatrix} x - \begin{bmatrix} y \\ \sqrt{\mu}g \end{bmatrix} \right\|^2 \\ &= \|\tilde{A}x - \tilde{y}\|^2\end{aligned}$$

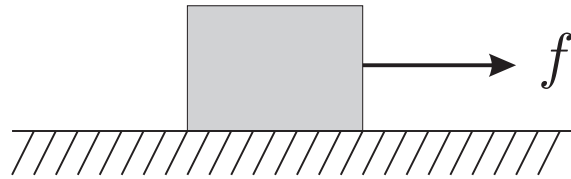
where

$$\tilde{A} = \begin{bmatrix} A \\ \sqrt{\mu}F \end{bmatrix} \quad \tilde{y} = \begin{bmatrix} y \\ \sqrt{\mu}g \end{bmatrix}$$

hence solution is (assuming \tilde{A} full rank)

$$\begin{aligned}x &= \tilde{A}^\dagger \tilde{y} \\ &= (A^T A + \mu F^T F)^{-1} (A^T y + \mu F^T g)\end{aligned}$$

example: forces on mass



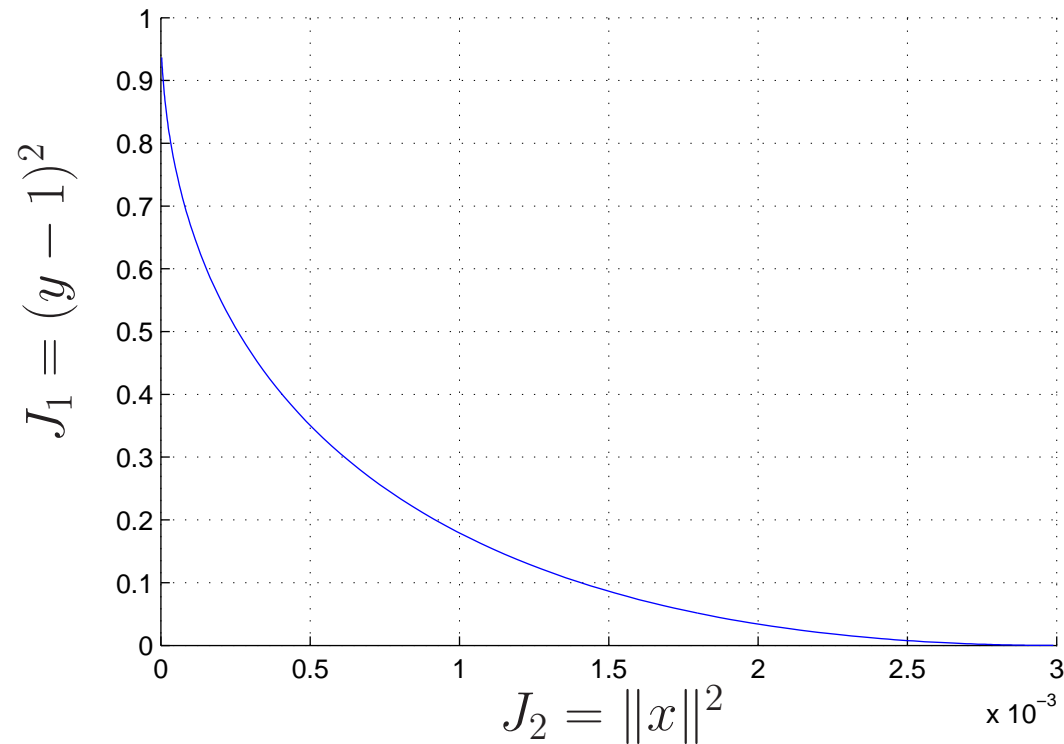
- unit mass at rest, subject to forces x_i for $i - 1 \leq t \leq i$, for $i = 1, \dots, 10$
- $y \in \mathbb{R}$ is position at $t = 10$; $y = a^T x$ where $a \in \mathbb{R}^{10}$
- $J_1 = (y - 1)^2$ (final position error squared)
- $J_2 = \|x\|^2$ (sum of squares of forces)

weighted sum objective: $(a^T x - 1)^2 + \mu \|x\|^2$

optimal x :

$$x = (aa^T + \mu I)^{-1} a$$

optimal trade-off curve



- upper left corner of optimal trade-off curve corresponds to $x = 0$
- bottom right corresponds to input that yields $y = 1$, i.e., $J_1 = 0$