

14. Observability

- Estimating the initial state
- Observability
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- Least-squares observers
- The observability ellipsoid
- Infinite time
- Computing observability
- Example: one mass attached to two springs
- Estimating other states
- Example: estimating other states

The Key Points of This Section

- once we know $u(0), \dots, u(T-1)$ and $y(0), \dots, y(T-1)$, we just have a linear equation relating the data to the initial state $x(0)$
- so we can use least squares to estimate it
- and we can compute the *estimation ellipsoid*
- which tells how sensitive the estimate is to errors
- the infinite-time case is easy to compute via a *Lyapunov equation*
- which gives a *practical measure* of observability

Estimating the Initial State

discrete-time system

$$\begin{aligned}x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}$$

which means

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(T-1) \end{bmatrix} = \begin{bmatrix} D & & & & \\ CB & D & & & \\ CAB & CB & D & & \\ \vdots & & & \ddots & \\ CA^{T-2}B & CA^{T-3}B & \dots & CB & D \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(T-1) \end{bmatrix} + \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{T-1} \end{bmatrix} x(0)$$

we'd like to estimate $x(0)$; ask *estimation* questions

- given $u(0), \dots, u(T-1)$ and $y(0), \dots, y(T-1)$, find $x(0)$
- find *all* $x(0)$ consistent with measured data
- if there is no exactly consistent $x(0)$, find an approximate one

estimating the initial state

this is just

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(T-1) \end{bmatrix} = P_T \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(T-1) \end{bmatrix} + J_T x(0)$$

where

$$P_T = \begin{bmatrix} D & & & & \\ CB & D & & & \\ CAB & CB & D & & \\ \vdots & & & \ddots & \\ CA^{T-2}B & CA^{T-3}B & \dots & CB & D \end{bmatrix} \quad J_T = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{T-1} \end{bmatrix}$$

- P_T is block Toeplitz; when $x(0) = 0$ it maps the input sequence to the output sequence
- J_T maps initial state to output sequence

estimating the initial state

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(T-1) \end{bmatrix} - P_T \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(T-1) \end{bmatrix} = J_T x(0)$$

- this is the usual estimation problem of the form ' $y_{\text{meas}} = Ax$ ', where y_{meas} is the LHS of the equation above
 x is $x(0)$
 A is J_T
- we need
 - more measurements than unknowns; need J_T to be skinny
 - $\text{null}(J_T) = \{0\}$ i.e. J_T must have full rank
 otherwise we will have *ambiguity* in solution for $x(0)$

observability

- if $z_1, z_2 \in \mathbb{R}^n$, and $z_1 - z_2 \in \text{null}(J_T)$, then initial states $x(0) = z_1$ and $x(0) = z_2$ are *indistinguishable* given $u(0), \dots, u(T-1)$ and $y(0), \dots, y(T-1)$.
- if $T \geq n$, then by the *Cayley-Hamilton theorem*,

$$\text{null}(J_T) = \text{null} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{T-1} \end{bmatrix} = \text{null} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- which means: if we can determine $x(0)$ from $u(0), \dots, u(T-1), y(0), \dots, y(T-1)$, then we can determine it from $u(0), \dots, u(n-1), y(0), \dots, y(n-1)$
- $\text{null}(J_n)$ is called the *unobservable subspace*
 J_n is called the *observability matrix*
 if $\text{null}(J_n) = \{0\}$ we call the system *observable*

example: observability

$$x(t+1) = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} x(t)$$

then

$$J_n = J_2 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 5 \end{bmatrix}$$

$\text{rank}(J_2) = 2$ so the system is observable

Least-Squares Observers

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(T-1) \end{bmatrix} - P_T \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(T-1) \end{bmatrix} = J_T x(0)$$

so we can estimate $x(0)$ by

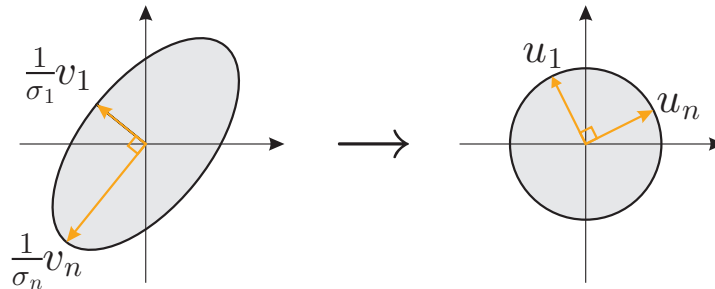
$$x_{\text{est}}(0) = J_T^\dagger \left(\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(T-1) \end{bmatrix}_{\text{measured}} - P_T \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(T-1) \end{bmatrix}_{\text{measured}} \right)$$

this gives the exact solution if one exists

Observability Ellipsoid

when $u(0) = u(1) = \dots = u(T-1) = 0$, we have
$$\begin{bmatrix} y(0) \\ \vdots \\ y(T-1) \end{bmatrix} = J_T x(0)$$

which states give *output energy*
$$\sum_{t=0}^{T-1} \|y(t)\|^2 \leq 1$$



- semi-axis directions given by right-singular vectors of J_T
semi-axes lengths given by $\frac{1}{\sigma_i(J_T)}$

observability ellipsoid

when the input is zero,

$$\begin{aligned} \sum_{t=0}^{T-1} \|y(t)\|^2 &= \begin{bmatrix} y(0) \\ \vdots \\ y(T-1) \end{bmatrix}^T \begin{bmatrix} y(0) \\ \vdots \\ y(T-1) \end{bmatrix} \\ &= x(0)^T J_T^T J_T x(0) \end{aligned}$$

observability ellipsoid is

$$\left\{ x \in \mathbb{R}^n \mid x^T J_T^T J_T x \leq 1 \right\}$$

- eigenvalues of $J_T^T J_T$ are squared singular values of J_T
eigenvectors of $J_T^T J_T$ are right singular vectors of J_T
- long axes: *weakly observable* directions
short axes: *strongly observable* directions

observability ellipsoid

let

$$\begin{aligned}
 V_T &= J_T^T J_T \\
 &= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{T-1} \end{bmatrix}^T \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{T-1} \end{bmatrix} \\
 &= \sum_{k=0}^{T-1} (A^T)^k C^T C A^k
 \end{aligned}$$

so if $t \geq s$ then $V_t \geq V_s$, i.e.,

- observability ellipsoid is smaller at larger t
- states are more observable given more data

observability ellipsoid over infinite time

to measure *practical observability*, look at infinite-time case

$$V = \sum_{k=0}^{\infty} (A^T)^k C^T C A^k$$

sum converges if $\rho(A) < 1$; then V satisfies

$$V - A^T V A = C^T C$$

called observability Lyapunov equation

computing observability

just like controllability:

- solution of Lyapunov equation $V - A^TVA = C^TC$ is easy; just linear equations
solution is unique if $\rho(A) < 1$

- infinite time observability ellipsoid is

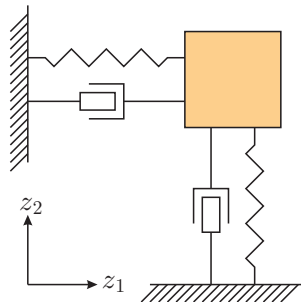
$$\left\{ x \in \mathbb{R}^n \mid x^TVx \leq 1 \right\}$$

large eigenvalues means corresponding direction is strongly observable

- V is called *observability Gramian*

example: initial state observation

- unit mass on frictionless surface
- connected to two springs arranged perpendicularly
- applied force $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$



linearized equations of motion are

$$\begin{aligned} \ddot{z}_1 + b_1\dot{z}_1 + k_1z_1 &= u_1 \\ \ddot{z}_2 + b_2\dot{z}_2 + k_2z_2 &= u_2 \end{aligned}$$

example: state equations

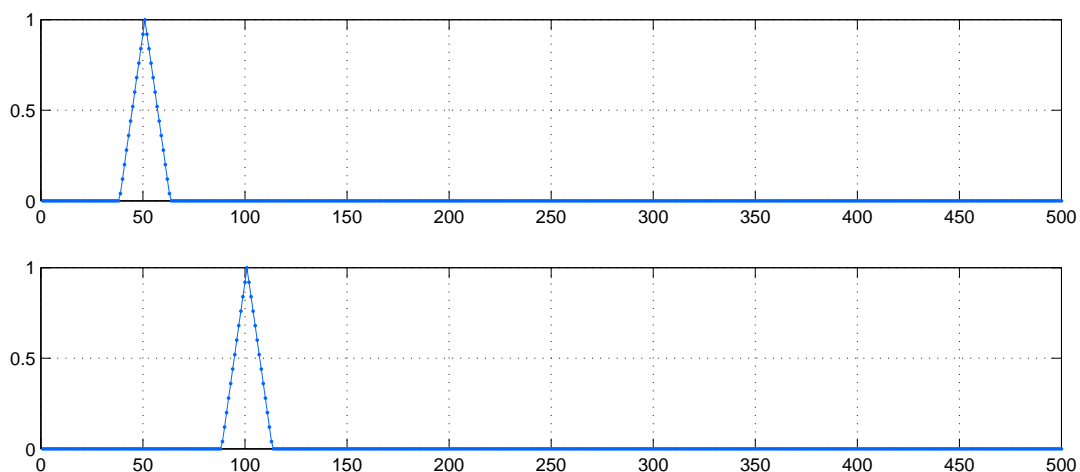
- spring constants $k_1 = 0.1$, $k_2 = 0.5$
- damping constants $b_1 = 0.4$, $b_2 = 0.3$
- states $x(t) = [z_1(t) \ z_2(t) \ \dot{z}_1(t) \ \dot{z}_2(t)]^T$

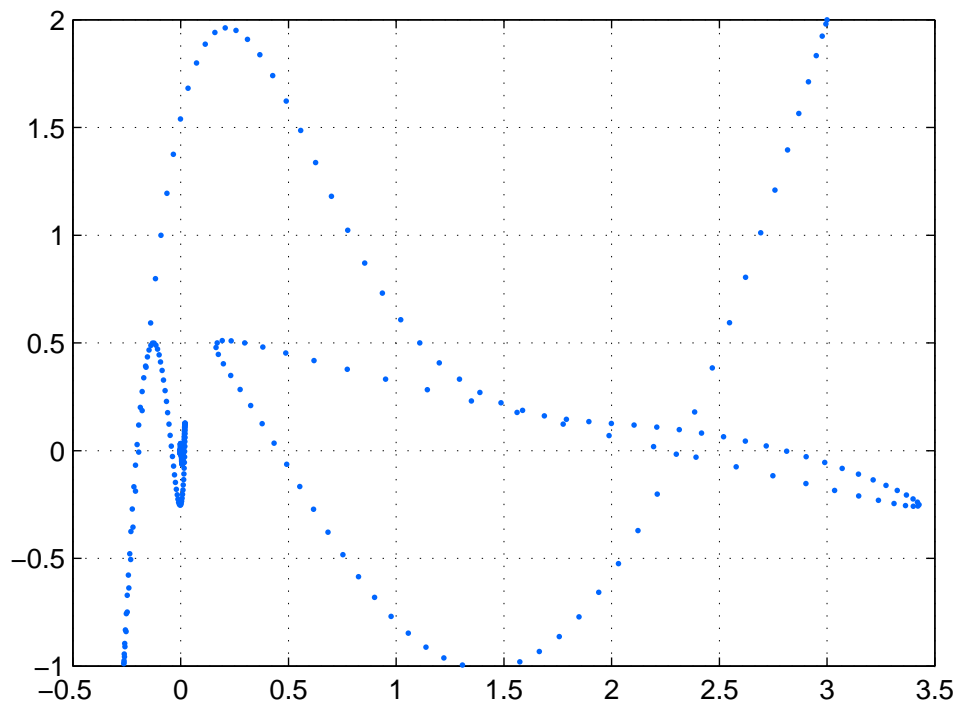
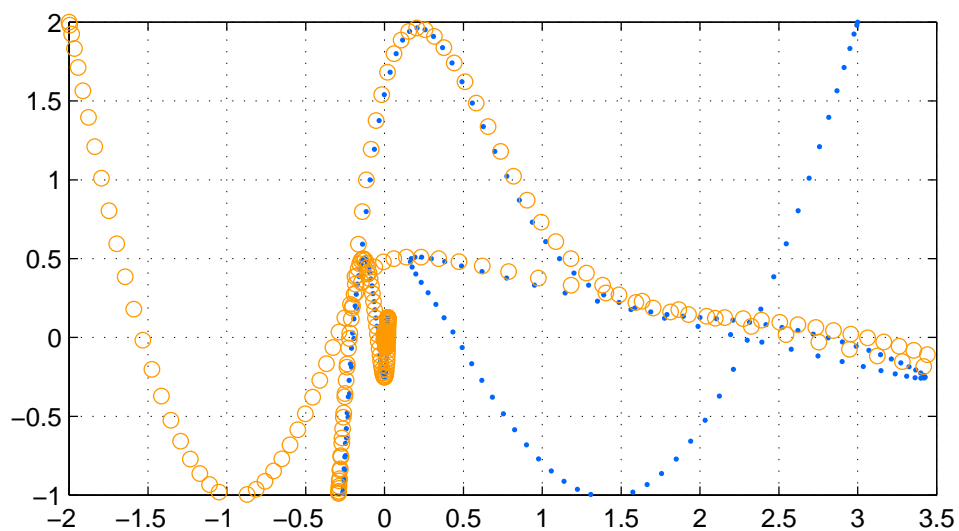
$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1 & 0 & -b_1 & 0 \\ 0 & -k_2 & 0 & -b_2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} x(t)$$

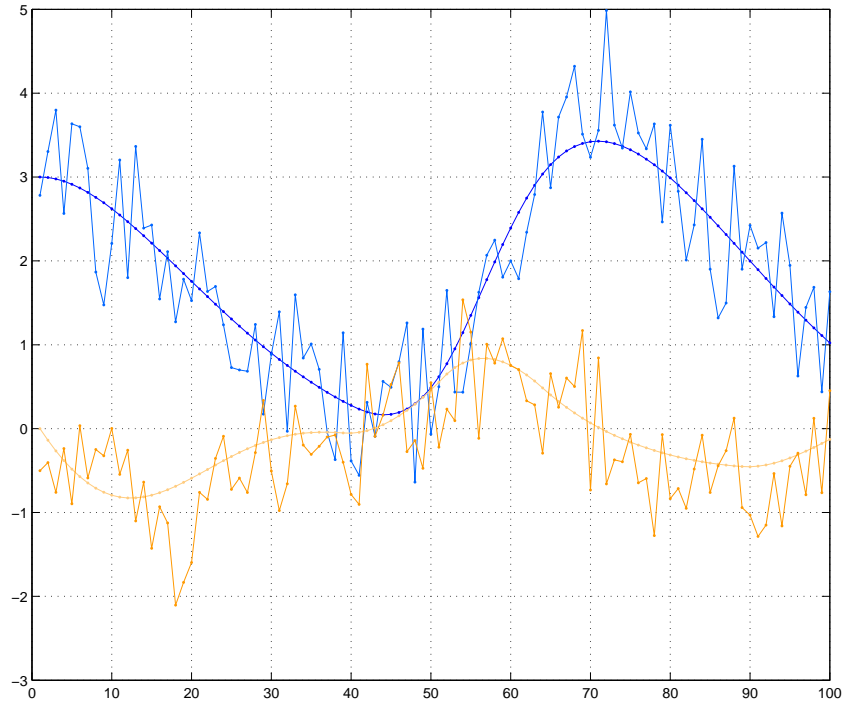
discretize, sampling period $h = 0.2$ gives

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

example: input signals

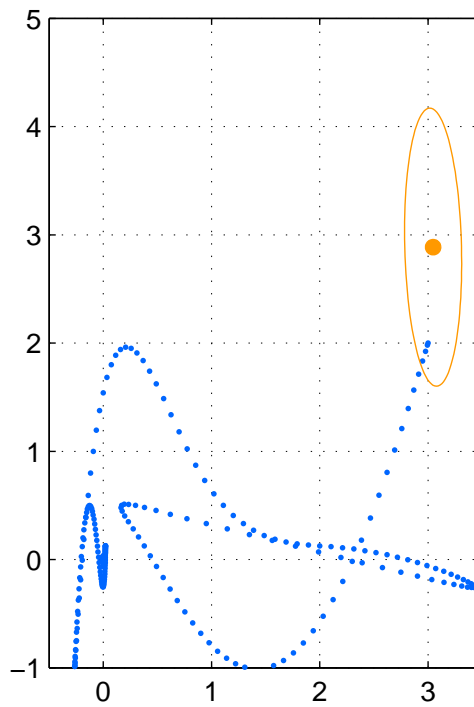
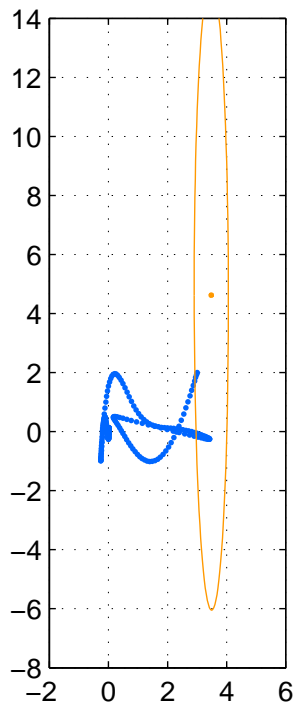
example: trajectoryinitial state $x(0) = [3 \ 2 \ 0 \ 0]^T$ **example: fading effect of initial conditions**initial state $x(0) = [3 \ 2 \ 0 \ 0]^T$ another initial state $[-2 \ 2 \ 0 \ 0]^T$ 

example: noisy measurements



example: estimate and error ellipsoids

after 3 measurements and after 20 measurements
 estimate and worst-case ellipsoid with unit norm noise



estimating other states

we can also use least squares to estimate later states $x(t)$

once we have an estimate of $x(0)$, then

$$x_{\text{est}}(t) = A^t x_{\text{est}}(0) + \sum_{\tau=0}^{t-1} A^{(t-1-\tau)} B u(\tau)$$

we'll analyze the error properties in Engr207b

example: estimating other states

Estimating all states based on a 20-measurement estimate of $x(0)$.

