EE263 Review session 3

- Eigenvalues / eigenvectors
- Symmetric matrices
- Quadratic forms
- Positive definiteness
- Examples

Eigenvalues and eigenvectors

For $A \in \mathbb{R}^{n \times n}$, there exists nonzero $v \in \mathbb{C}^n$ such that

$$\boxed{Av = \lambda v}$$

any such v is called an eigenvector of A associated with eigenvalue λ

how to compute

- $(\lambda I - A)$ has nonzero null space, thus

$$\det(\lambda I - A) = 0$$

  which is a polynomial of order n

- the following matlab command finds eigenvectors and eigenvalues

```
[V D] = eig(A);
```

examples

- consider $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

- λ makes $(\lambda I - A)$ rank deficient,
  $\Rightarrow \lambda_1 = 3$ and $\lambda_2 = -1$

- $v \in \mathrm{null}(\lambda I - A)$,
  $\Rightarrow v_1 = [1\ 1]^T$ and $v_2 = [1\ \ -1]^T$

- what if A is rank-deficient already?
  for example, $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

examples

- what are the eigenvalues of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$?

- what are the eigenvalues of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$?

- what are the eigenvalues of $A = \begin{bmatrix} 1 & 0 & 5 & 2 & 4 \\ 9 & 3 & 3 & 1 & 5 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 5 & 7 & 0 \\ 0 & 0 & 9 & 8 & 2 \end{bmatrix}$?

Stability of linear dynamical systems

- a discrete time linear dynamical system

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) \\
y(k) &= Cx(k) + Du(k)
\end{aligned}
$$

is stable if and only if

$$\max |\lambda(A)| < 1$$

- a continuous time linear dynamical system

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
$$

is stable if and only if

$$\max \operatorname{Re} \lambda(A) < 0$$

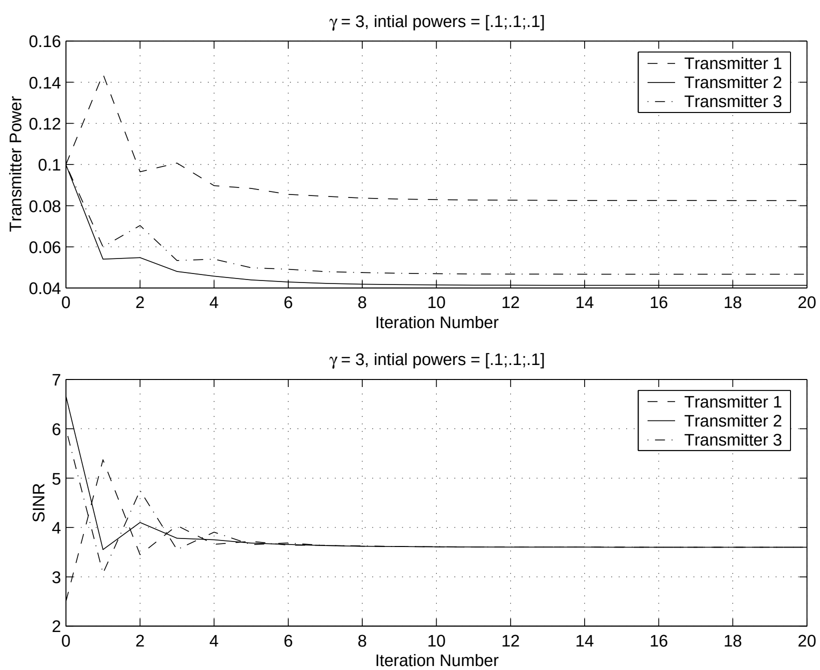
Homework revisited

a simple power control algorithm for a wireless network (hw set#1)

- we discribed a power control algorithm as a linear dynamical system as,

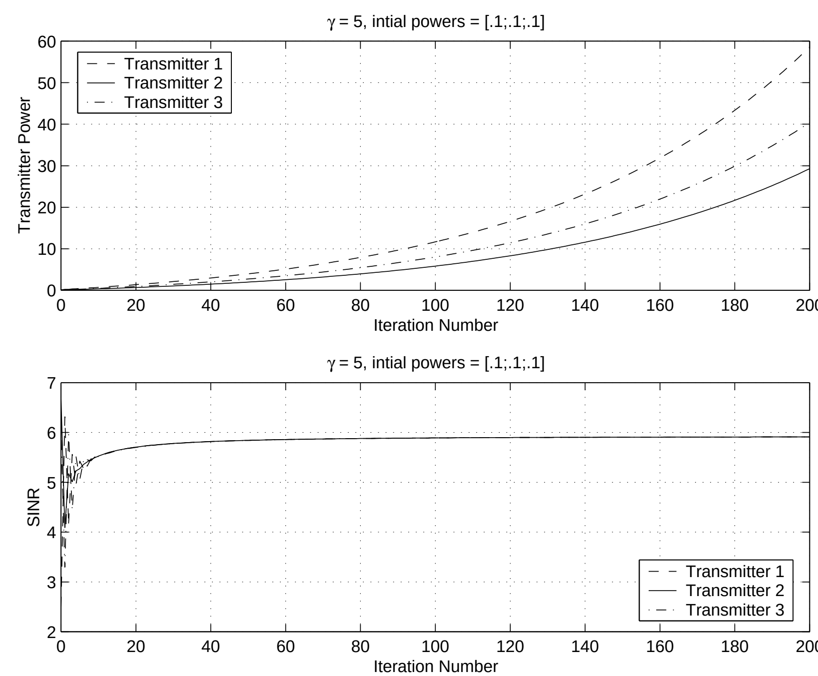
$$p(t+1) = Ap(t) + b \qquad A_{ij} = \begin{cases} 0 & \text{if } i = j \\ \alpha\gamma G_{ij}/G_{ii} & \text{otherwise} \end{cases}$$

- and simulated two cases, $\gamma = 3$ and $\gamma = 5$.



$\gamma = 3$

$\max|\lambda(A)| = 0.608$



$\gamma = 5$

$\max|\lambda(A)| = 1.014$

Eigenvalues of symmetric matrices

for symmetric matrices

- eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ are *real*
- eigenvectors $\{q_1, \ldots, q_n\}$ form an *orthonormal basis* for $\mathbb{R}^n$
- thus diagonalized by an orthogonal matrix

$$U^T A U = \Lambda$$

where

$$U = \begin{bmatrix} q_1 & q_2 & \ldots & q_n \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

- A is written as

$$A = U \Lambda U^T = \sum_{i=1}^{n} \lambda_i q_i q_i^T$$

Example

torque and angular motion

- angular acceleration of a rigid body is a linear function of the applied torque

$$\begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix}$$

- *principal axis:* the direction along which the applied torque and the resulting angular acceleration align
- *principal moment of inertia:* the scale factor for such cases

Quadratic forms

- if A is a symmetric matrix, the quadratic function $f(x) = x^T A x$ which maps $\mathbb{R}^n$ to $\mathbb{R}$ is called a *quadratic form.* for example,

$$x^T A x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{13}x_1x_3$$

- We can bound $x^T A x$

$$\lambda_{min} x^T x \leq x^T A x \leq \lambda_{max} x^T x$$

Example

quadratic optimization problems

- we can solve the following problem easily

$$\begin{array}{ll} \text{maximize} & x^T A x \\ \text{subject to} & x^T x \leq 1 \end{array}$$

- what about the following?

$$\begin{array}{ll} \text{maximize} & \|Ax\|_2 \\ \text{subject to} & x^T x \leq 1 \end{array}$$

Homework revisited

the final position/velocity of a mass (hw set#1)

- maximizing the final position boils down to

$$\begin{array}{ll} \text{maximize} & a_1^T x \\ \text{subject to} & x^T x \leq 1 \end{array}$$

  which is equivalent to

$$\begin{array}{ll} \text{maximize} & x^T (a_1 a_1^T) x \\ \text{subject to} & x^T x \leq 1 \end{array}$$

- eigenvalues of $a_1 a_1^T$?

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Positive definite matrices

positive definiteness

- A is positive definite if $x^T Ax > 0$ for all nonzero x
- A is positive semidefinite if $x^T Ax \geq 0$ for all x

Schur complement

- Suppose A is invertible, then

$$\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^T A x + 2x^T B y + y^T D y$$
$$= (x + A^{-1}By)^T A (x + A^{-1}By) + y^T (D - B^T A^{-1} B) y$$

- $D - B^T A^{-1} B$ is called the *Schur complement*

Examples

optimization over partial variables

- find x that minimizes the following quadratic form

$$
\begin{aligned}
f(x,y) &= \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= x^T A x + 2x^T B y + y^T D y \\
&= (x + A^{-1}By)^T A (x + A^{-1}By) + y^T (D - B^T A^{-1} B) y
\end{aligned}
$$

- $\min_x f(x,y) = y^T (D - B^T A^{-1} B) y$

feasibility of a polynomial inequality

- is the following polynomial inequality feasible?

$$f(x) = 4x_1^4 + 4x_1^3x_2 - 7x_1^2x_2^2 - 2x_1x_2^3 + 10x_2^4 < 0$$

- in quadratic form,

$$f(x) = \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}^T \begin{bmatrix} 4 & -\gamma & 2 \\ -\gamma & 10 & -1 \\ 2 & -1 & -7+2\gamma \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}$$

- if $\begin{bmatrix} 4 & -\gamma & 2 \\ -\gamma & 10 & -1 \\ 2 & -1 & -7+2\gamma \end{bmatrix} > 0$, such x_1 and x_2 do not exist.

- infeasibility certificate: $\gamma = 6$

$$\begin{bmatrix} 4 & -6 & 2 \\ -6 & 10 & -1 \\ 2 & -1 & 5 \end{bmatrix} > 0$$