

# 6. Symmetric and Positive Matrices

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# Eigenvalues of Symmetric Matrices

a matrix  $A \in \mathbb{R}^{n \times n}$  is called *symmetric* if  $A = A^T$

## eigenvalues

if  $A$  is symmetric, then the eigenvalues of  $A$  are real

to show this, suppose  $x$  is an eigenvector of  $A$  with eigenvalue  $\lambda \in \mathbb{C}$ . Then  $Ax = \lambda x$  and  $x \neq 0$

we will show that  $\lambda = \bar{\lambda}$ . we know that

$$\bar{x}^T Ax = \lambda \bar{x}^T x = \lambda \sum_{i=1}^n |x_i|^2$$

also

$$\bar{x}^T Ax = (\overline{Ax})^T x = \bar{\lambda} \bar{x}^T x = \bar{\lambda} \sum_{i=1}^n |x_i|^2$$

since  $\|x\| \neq 0$ , we have  $\lambda = \bar{\lambda}$

# Eigenvectors of Symmetric Matrices

for symmetric matrices:

eigenvectors corresponding to distinct eigenvalues are orthogonal

suppose  $Ax_1 = \lambda_1 x_1$  and  $Ax_2 = \lambda_2 x_2$ . Then

$$\begin{aligned}x_1^T Ax_2 &= (Ax_1)^T x_2 \\ &= \lambda_1 x_1^T x_2\end{aligned}$$

and also

$$x_1^T Ax_2 = \lambda_2 x_1^T x_2$$

therefore

$$(\lambda_1 - \lambda_2)x_1^T x_2 = 0$$

and since  $\lambda_1 \neq \lambda_2$  we must have  $x_1^T x_2 = 0$

- if  $A$  is symmetric and has  $n$  distinct eigenvalues, then its eigenvectors form an *orthogonal basis* for the vector space  $\mathbb{R}^n$

## eigenvectors

**fact:** every  $n \times n$  symmetric matrix  $A$  has  $n$  eigenvectors

$$\{q_1, \dots, q_n\}$$

which form an *orthonormal basis* for  $\mathbb{R}^n$ .

## in matrix notation

$$U = [q_1 \ q_2 \ \dots \ q_n] \qquad \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

then, since  $q_i$  are eigenvectors, we have  $AU = U\Lambda$  and so

$$U^{-1}AU = \Lambda$$

i.e. the columns of  $U$  are an orthonormal basis which *diagonalizes*  $A$

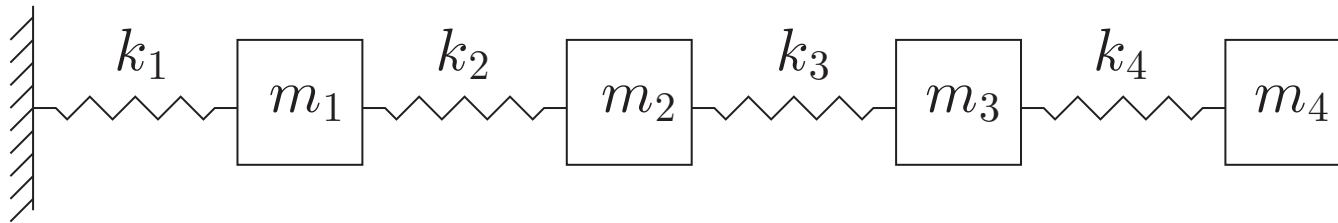
also  $U$  is an orthogonal matrix, so  $U^{-1} = U^T$ .

## example: symmetric matrices in mechanical systems

for linear undamped mechanical systems (e.g. elasticity), the equations of motion are

$$M\ddot{x}(t) + Kx(t) = 0$$

where  $M$  and  $K$  are symmetric matrices. for example:



for this system,

$$M = I \quad K = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

with  $q = U^{-1}x$  and  $K = U\Lambda U^{-1}$ , we have

$$\ddot{q}(t) + \Lambda q(t) = 0$$

and the motion decomposes into 4 *normal modes*

# Quadratic Forms

if  $A$  is a symmetric matrix, the quadratic function  $f(x) = x^T A x$  which maps  $\mathbb{R}^n$  to  $\mathbb{R}$  is called a *quadratic form*. for example,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 0 \\ -1 & 0 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x^2 + 4xy - 2xz - 3y^2 + 7z^2$$

## positive definite matrices

the symmetric matrix  $A$  is called *positive definite*, written  $A > 0$ , if

$$x^T A x > 0 \quad \text{for all nonzero } x \in \mathbb{R}^n$$

## examples

- $cI$  is positive definite if and only if  $c > 0$ .
- $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$  is positive definite if and only if  $a > 0$ ,  $c > 0$  and  $b^2 < ac$

## quadratic forms and eigenvalues

suppose  $A$  is symmetric with eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . then

$$\lambda_1 \|x\|^2 \geq x^T A x \geq \lambda_n \|x\|^2 \quad \text{for all } x \in \mathbb{R}^n$$

to see this, let  $\{q_1, \dots, q_n\}$  be orthonormal eigenvectors of  $A$  with the above eigenvalues. then

$$\begin{aligned} x^T A x &= \sum_{i=1}^n \lambda_i |q_i^T x|^2 \\ &\geq \lambda_n \sum_{i=1}^n |q_i^T x|^2 = \lambda_n \|x\|^2 \end{aligned}$$

### notes

- picking  $x = q_n$  gives  $x^T A x = \lambda_n \|x\|^2$  and so the inequality is *tight*
- this gives the following test for positive definiteness:

$$A > 0 \quad \iff \quad \lambda_{\min}(A) > 0$$

## notation for symmetric matrices

- $A$  is called *positive semidefinite* if

$$x^T A x \geq 0 \quad \text{for all } x \in \mathbb{R}^n$$

- we write  $A < 0$  to mean  $-A > 0$ . In this case, we say  $A$  is *negative definite*.
- we write  $A > B$  to mean  $A - B > 0$ .
- similar notation for semidefiniteness.
- a matrix which is neither positive semidefinite nor negative semidefinite is called *indefinite*.

## properties of positive definite matrices

- *addition of positive matrices*

$$A > 0 \text{ and } B > 0 \quad \implies \quad A + B > 0$$

- *block diagonal matrices*

$$A > 0 \text{ and } B > 0 \quad \iff \quad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} > 0$$

- *invertibility*

$$A > 0 \quad \implies \quad A \text{ is nonsingular}$$

- *scaling*

$$A, B \geq 0 \quad \text{and} \quad \lambda, \mu \geq 0 \quad \implies \quad \lambda A + \mu B \geq 0$$

## matrix square roots

if  $A$  is positive semidefinite, then

$$\begin{aligned} A &= U\Lambda U^T \\ &= U\Lambda^{\frac{1}{2}}U^T U\Lambda^{\frac{1}{2}}U^T \\ &= (U\Lambda^{\frac{1}{2}}U^T)^2 \end{aligned}$$

where

$$\Lambda^{\frac{1}{2}} = \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n})$$

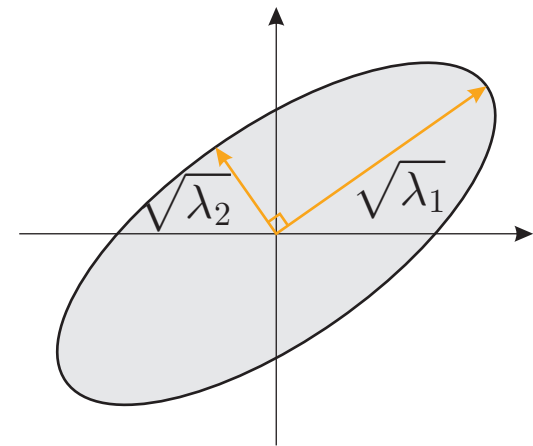
the matrix  $U\Lambda^{\frac{1}{2}}U^T$  is called the *square root* of  $A$

# Ellipsoids

an ellipsoid is a sphere stretched in orthogonal directions called the *principal semiaxes*

every positive definite matrix has a corresponding ellipsoid

$$\text{ellipse}(B) = \{ x \in \mathbb{R}^n \mid x^T B^{-1} x \leq 1 \}$$



## Notes

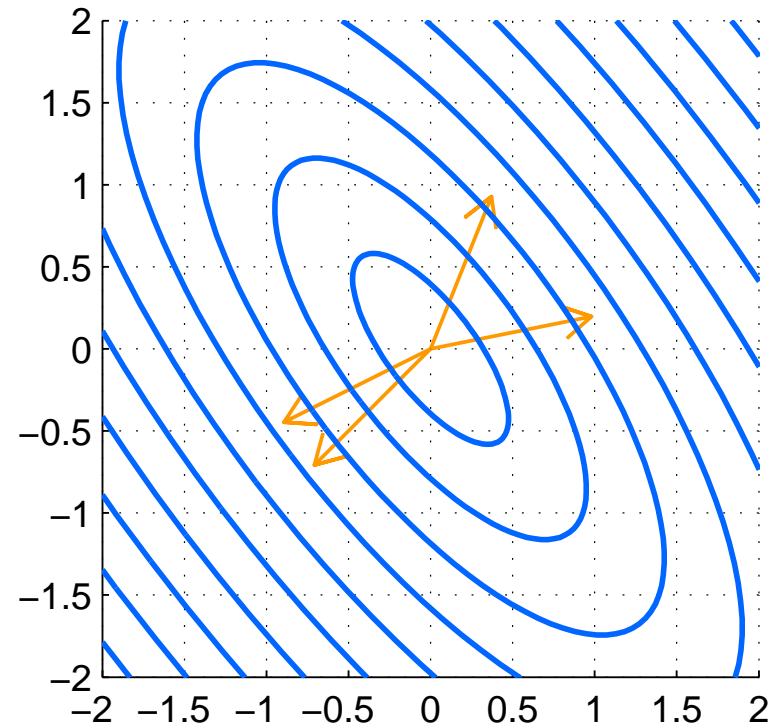
- $B \in \mathbb{R}^{n \times n}$ ,  $B = B^T$ ,  $B > 0$ .
- semiaxis lengths:  $\sqrt{\lambda_i}$ , where  $\lambda_i$  are eigenvalues of  $B$
- semiaxis directions: eigenvectors of  $B$
- $B$  is positive definite if and only if  $B^{-1}$  is, and they have the same eigenvectors.  
using  $B$  instead of  $B^{-1}$  replaces  $\sqrt{\lambda_i}$  by  $\frac{1}{\sqrt{\lambda_i}}$ .

## example: navigation

here  $A \in \mathbb{R}^{4 \times 2}$  with

$$A = \begin{bmatrix} b_1^T \\ b_2^T \\ b_3^T \\ b_4^T \end{bmatrix}$$

and  $y = Ax$ . Each  $b_i$  is a unit vector.



The contours of  $\|Ax\|$  are

$$\|Ax\| \leq c \quad \Leftrightarrow \quad x \in \text{ellipse}(c^2(A^T A)^{-1})$$

the sensors are most sensitive to the component of  $x$  along the *short axis* of the ellipsoid