6. Symmetric and Positive Matrices

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Eigenvalues of Symmetric Matrices

a matrix $A \in \mathbb{R}^{n \times n}$ is called *symmetric* if $A = A^T$

eigenvalues

> if A is symmetric, then the eigenvalues of A are real

to show this, suppose x is an eigenvector of A with eigenvalue $\lambda \in \mathbb{C}$. Then $Ax = \lambda x$ and $x \neq 0$

we will show that $\lambda = \bar{\lambda}$. we know that

$$\bar{x}^T A x = \lambda \bar{x}^T x = \lambda \sum_{i=1}^{n} |x_i|^2$$

also

$$\bar{x}^T A x = (\overline{Ax})^T x = \bar{\lambda} \bar{x}^T x = \bar{\lambda} \sum_{i=1}^{n} |x_i|^2$$

since $\|x\| \neq 0$, we have $\lambda = \bar{\lambda}$

Eigenvectors of Symmetric Matrices

for symmetric matrices:

eigenvectors corresponding to distinct eigenvalues are orthogonal

suppose $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$. Then

$$\begin{aligned} x_1^T A x_2 &= (Ax_1)^T x_2 \\ &= \lambda_1 x_1^T x_2 \end{aligned}$$

and also

$$x_1^T A x_2 = \lambda_2 x_1^T x_2$$

therefore

$$(\lambda_1 - \lambda_2) x_1^T x_2 = 0$$

and since $\lambda_1 \neq \lambda_2$ we must have $x_1^T x_2 = 0$

- if A is symmetric and has n distinct eigenvalues, then its eigenvectors form an *orthogonal basis* for the vector space $\mathbb{R}^n$

eigenvectors

fact: every $n \times n$ symmetric matrix A has n eigenvectors

$$\{q_1, \dots, q_n\}$$

which form an *orthonormal basis* for $\mathbb{R}^n$.

in matrix notation

$$U = \begin{bmatrix} q_1 & q_2 & \dots & q_n \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

then, since q_i are eigenvectors, we have $AU = U\Lambda$ and so

$$\boxed{U^{-1} A U = \Lambda}$$

i.e. the columns of U are an orthonormal basis which *diagonalizes* A

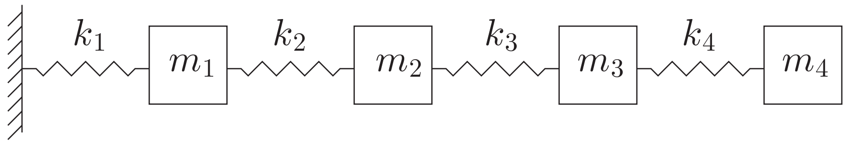
also U is an orthogonal matrix, so $U^{-1} = U^T$.

example: symmetric matrices in mechanical systems

for linear undamped mechanical systems (e.g. elasticity), the equations of motion are

$$M\ddot{x}(t) + Kx(t) = 0$$

where M and K are symmetric matrices. for example:



for this system,

$$M = I \qquad K = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

with $q = U^{-1}x$ and $K = U\Lambda U^{-1}$, we have

$$\ddot{q}(t) + \Lambda q(t) = 0$$

and the motion decomposes into 4 *normal modes*

Quadratic Forms

if A is a symmetric matrix, the quadratic function $f(x) = x^T A x$ which maps $\mathbb{R}^n$ to $\mathbb{R}$ is called a *quadratic form.* for example,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 0 \\ -1 & 0 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x^2 + 4xy - 2xz - 3y^2 + 7z^2$$

positive definite matrices

the symmetric matrix A is called *positive definite*, written $A > 0$, if

$$x^T A x > 0 \qquad \text{for all nonzero } x \in \mathbb{R}^n$$

examples

- cI is positive definite if and only if $c > 0$.
- $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is positive definite if and only if $a > 0$, $c > 0$ and $b^2 < ac$

quadratic forms and eigenvalues

suppose A is symmetric with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. then

$$\lambda_1 \|x\|^2 \geq x^T A x \geq \lambda_n \|x\|^2 \qquad \text{for all } x \in \mathbb{R}^n$$

to see this, let $\{q_1, \ldots, q_n\}$ be orthonormal eigenvectors of A with the above eigenvalues. then

$$\begin{aligned} x^T A x &= \sum_{i=1}^n \lambda_i |q_i^T x|^2 \\ &\geq \lambda_n \sum_{i=1}^n |q_i^T x|^2 = \lambda_n \|x\|^2 \end{aligned}$$

notes

- picking $x = q_n$ gives $x^T A x = \lambda_n \|x\|^2$ and so the inequality is *tight*
- this gives the following test for positive definiteness:

$$A > 0 \qquad \iff \qquad \lambda_{\text{min}}(A) > 0$$

notation for symmetric matrices

- A is called *positive semidefinite* if

$$x^T A x \geq 0 \qquad \text{for all } x \in \mathbb{R}^n$$

- we write $A < 0$ to mean $-A > 0$. In this case, we say A is *negative definite*.
- we write $A > B$ to mean $A - B > 0$.
- similar notation for semidefiniteness.
- a matrix which is neither positive semidefinite nor negative semidefinite is called *indefinite*.

properties of positive definite matrices

- *addition of positive matrices*

$$A > 0 \text{ and } B > 0 \qquad \Longrightarrow \qquad A + B > 0$$

- *block diagonal matrices*

$$A > 0 \text{ and } B > 0 \qquad \Longleftrightarrow \qquad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} > 0$$

- *invertibility*

$$A > 0 \qquad \Longrightarrow \qquad A \text{ is nonsingular}$$

- *scaling*

$$A, B \geq 0 \quad \text{and} \quad \lambda, \mu \geq 0 \qquad \Longrightarrow \qquad \lambda A + \mu B \geq 0$$

matrix square roots

if A is positive semidefinite, then

$$\begin{aligned} A &= U \Lambda U^T \\ &= U \Lambda^{\frac{1}{2}} U^T U \Lambda^{\frac{1}{2}} U^T \\ &= \big(U \Lambda^{\frac{1}{2}} U^T \big)^2 \end{aligned}$$

where

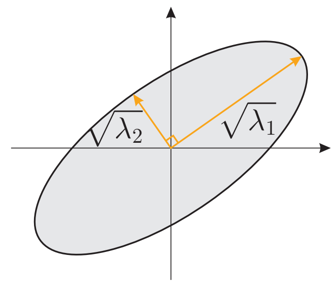
$$\Lambda^{\frac{1}{2}} = \operatorname{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n})$$

the matrix $U \Lambda^{\frac{1}{2}} U^T$ is called the *square root* of A

Ellipsoids

an ellipsoid is a sphere stretched in orthogonal directions called the *principal semiaxes*

> every positive definite matrix has a corresponding ellipsoid



$$\text{ellipse}(B) = \left\{ x \in \mathbb{R}^n \mid x^T B^{-1} x \leq 1 \right\}$$

Notes

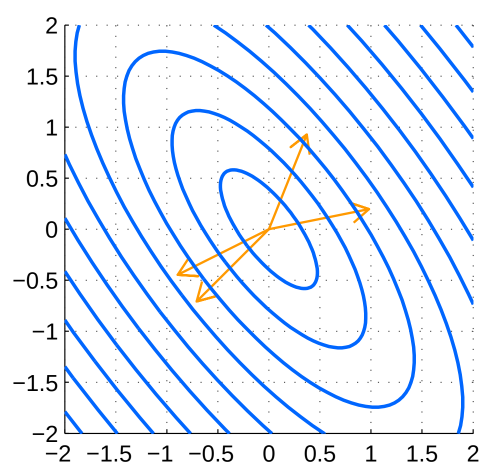
- $B \in \mathbb{R}^{n \times n}$, $B = B^T$, $B > 0$.
- semiaxis lengths: $\sqrt{\lambda_i}$, where λ_i are eigenvalues of B
- semiaxis directions: eigenvectors of B
- B is positive definite if and only if B^{-1} is, and they have the same eigenvectors. using B instead of B^{-1} replaces $\sqrt{\lambda_i}$ by $\frac{1}{\sqrt{\lambda_i}}$.

example: navigation

here $A \in \mathbb{R}^{4 \times 2}$ with

$$A = \begin{bmatrix} b_1^T \\ b_2^T \\ b_3^T \\ b_4^T \end{bmatrix}$$

and $y = Ax$. Each b_i is a unit vector.



The contours of $\|Ax\|$ are

$$\|Ax\| \leq c \quad \Leftrightarrow \quad x \in \text{ellipse}\big(c^2 (A^T A)^{-1}\big)$$

> the sensors are most sensitive to the component of x along the *short axis* of the ellipsoid