EE263 Prof. S. Boyd
Dec. 3–4 or Dec. 4–5, 2010.

Final exam

This is a 24 hour take-home final exam. Please turn it in at Bytes Cafe in the Packard building, 24 hours after you pick it up.

Please read the following instructions carefully.

- You may use any books, notes, or computer programs (*e.g.*, Matlab), but you may not discuss the exam with anyone until Dec. 6, after everyone has taken the exam. The only exception is that you can ask the TAs or Stephen Boyd for clarification, by emailing to the staff email address. We've tried pretty hard to make the exam unambiguous and clear, so we're unlikely to say much.
- Since you have 24 hours, we expect your solutions to be legible, neat, and clear. Do not hand in your rough notes, and please try to simplify your solutions as much as you can. We will deduct points from solutions that are technically correct, but much more complicated than they need to be.
- Please check your email a few times during the exam, just in case we need to send out a clarification or other announcement. It's unlikely we'll need to do this, but you never know.
- Attach the official exam cover page (available when you pick up or drop off the exam) to your exam, and assemble your solutions to the problems in order, *i.e.*, problem 1, problem 2, . . . , problem 7. Start each solution on a new page. Do not collect all plots or code (for example) at the end of the exam; plots for problem 3 (say) should be with your solution to problem 3.
- Please make a copy of your exam before handing it in. We have never lost one, but it might occur.
- When a problem involves some computation (say, using Matlab), we do not want just the final answers. We want a clear discussion and justification of exactly what you did, the Matlab source code that produces the result, and the final numerical result. Be sure to show us your verification that your computed solution satisfies whatever properties it is supposed to, at least up to numerical precision. For example, if you compute a vector x that is supposed to satisfy $Ax = b$ (say), show us the Matlab code that checks this, and the result. (This might be done with the Matlab code `norm(A*x-b)`; be sure to show us the result, which should be very small.) *We will not check your numerical solutions for you, in cases where there is more than one solution.*

- In the portion of your solutions where you explain the mathematical approach, you *cannot* refer to Matlab operators, such as the backslash operator. (You can, of course, refer to inverses of matrices, or any other standard mathematical construct.)

- Some of the problems are described in (what appears to be) a practical setting. *You do not need to understand anything about the application area to solve these problems.* We've taken special care to make sure *all* the information and math needed to solve the problem is given in the problem description.

- Some of the problems require you to download and run a Matlab file to generate the data needed. These files can be found at the URL

  `http://www.stanford.edu/class/ee263/final_10_11_mfiles/FILENAME`

  where you should substitute the particular filename (given in the problem) for `FILENAME`. *There are no links on the course web page pointing to these files, so you'll have to type in the whole URL yourself.*

- Please respect the honor code. Although we encourage you to work on homework assignments in small groups, *you cannot discuss the final with anyone*, with the exception of Stephen Boyd and the TAs, until everyone has taken it.

1. *Some attributes of a stable system.* This problem concerns the autonomous linear dynamical system $\dot{x} = Ax$, with $x(t) \in \mathbf{R}^n$, which we assume is stable (*i.e.*, all trajectories $x(t)$ converge to zero as $t \to \infty$).

   - *Peaking factor.* We define the peaking factor of the system as the largest possible value of $\|x(t+\tau)\|/\|x(t)\|$, for any nonzero trajectory x, any t, and any $\tau \geq 0$.
   - *Halving time.* We define the halving time of the system as the smallest $\tau \geq 0$ for which $\|x(t+\tau)\| \leq \|x(t)\|/2$ always holds, for all trajectories.
   - *Minimum decorrelation time.* We define the minimum decorrelation time as the smallest possible $\tau \geq 0$ for which $x(t+\tau) \perp x(t)$ can hold for some (nonzero) trajectory x. This is the smallest possible time the state can rotate 90°. (If $x(t+\tau) \perp x(t)$ never occurs for $\tau \geq 0$, then the minimum decorrelation time is $+\infty$.)

   (a) Explain how to find each of these quantities. Your method can involve some numerical simulation, such as a search over a (fine) grid of values of τ. You can assume that you do not need to search over τ greater than $\tau^{\max}$, where $\tau^{\max}$ is known.

   (b) Carry out your method for the specific case with

$$A = \begin{bmatrix} -1 & -5 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0.4 & -1 & -0.6 & -6 \\ 1 & 0 & 6 & 0 \end{bmatrix},$$

   with $\tau^{\max} = 10$. We'd like all quantities to an accuracy of around 0.01.

2. *Fleet modeling.* In this problem, we will consider model estimation for vehicles in a fleet. We collect input and output data at multiple time instances, for each vehicle in a fleet of vehicles:

$$x^{(k)}(t) \in \mathbf{R}^n, \quad y^{(k)}(t) \in \mathbf{R}, \quad t = 1, \ldots, T, \quad k = 1, \ldots, K.$$

Here k denotes the vehicle number, t denotes the time, $x^{(k)}(t) \in \mathbf{R}^n$ the input, and $y^{(k)}(t) \in \mathbf{R}$ the output. (In the general case the output would also be a vector; but for simplicity here we consider the scalar output case.)

While it does not affect the problem, we describe a more specific application, where the vehicles are airplanes. The components of the inputs might be, for example, the deflections of various control surfaces and the thrust of the engines; the output might be vertical acceleration. Airlines are required to collect this data, called FOQA data, for every commercial flight. (This description is not needed to solve the problem.)

We will fit a model of the form

$$y^{(k)}(t) \approx a^T x^{(k)}(t) + b^{(k)},$$

where $a \in \mathbf{R}^n$ is the (common) linear model parameter, and $b^{(k)} \in \mathbf{R}$ is the (individual) offset for the kth vehicle.

We will choose these to minimize the mean square error

$$E = \frac{1}{TK} \sum_{t=1}^{T} \sum_{k=1}^{K} \left(y^{(k)}(t) - a^T x^{(k)}(t) - b^{(k)} \right)^2.$$

(a) Explain how to find the model parameters a and $b^{(1)}, \ldots, b^{(K)}$.

(b) Carry out your method on the data given in `fleet_mod_data.m`. The data is given using cell arrays `X` and `y`. The columns of the $n \times T$ matrix `X{k}` are $x^{(k)}(1), \ldots, x^{(k)}(T)$, and the $1 \times T$ row vector `y{k}` contains $y^{(k)}(1), \ldots, y^{(k)}(T)$. Give the model parameters a and $b^{(1)}, \ldots, b^{(K)}$, and report the associated mean square error E. Compare E to the (minimum) mean square error E^{com} obtained using a common offset $b = b^{(1)} = \cdots = b^{(K)}$ for all vehicles.

    By examining the offsets for the different vehicles, suggest a vehicle you might want to have a maintenance crew check out. (This is a simple, straightforward question; we don't want to hear a long explanation or discussion.)

3. *System with level alarms.* A linear dynamical system evolves according to

$$\dot{x}(t) = Ax(t), \qquad y(t) = Cx(t),$$

where $x(t) \in \mathbf{R}^n$ is the state and $y(t) \in \mathbf{R}^p$ is the output at time t. You know A and C, but not $x(t)$ or $y(t)$, except as described below.

The output is monitored using level alarms with thresholds. These tell us when $y_i(t) \geq l_i$, where l_i is the threshold level for output component i. (The threshold levels l_i are known.)

You have alarm data over the time interval $[0, T]$, of the following format. For each output component $i = 1, \ldots, p$, you are given the (possibly empty) set of the intervals in $[0, T]$ over which $y_i(t) \geq l_i$.

We now consider the specific problem with

$$A = \begin{bmatrix} -0.9 & -4.2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix},$$

$T = 10$, $l_1 = l_2 = 1$, and alarm intervals given below:

$$\begin{array}{ll} y_1: & [0, 1.0195], \quad [3.0288, 4.0863], \quad [6.4176, 6.9723] \\ y_2: & [0.9210, 1.9402]. \end{array}$$

The problem is to find an upper bound on how large $\|x(T)\|$ can be, while being consistent with the given alarm data. We allow $+\infty$ as an answer here; this means that there are trajectories with arbitrarily large values of $\|x(T)\|$ that are consistent with the given alarm data. (We will deduct points for solutions that give bounds that are correct, but higher than they need to be.)

Give your bound on $\|x(T)\|$. If it is $+\infty$, explain. Of course, you must explain your method.

4. *Minimum energy control with delayed destination knowledge.* We consider a vehicle moving in $\mathbf{R}^2$, with dynamics

$$x(t+1) = Ax(t) + Bu(t), \quad p(t) = Cx(t), \quad t = 1, 2, \ldots, \qquad x(1) = 0,$$

where $x(t) \in \mathbf{R}^n$ is the state, $u(t) \in \mathbf{R}^m$ is the input, and $p(t) \in \mathbf{R}^2$ is the position of the vehicle, at time t. The matrices $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$ and $C \in \mathbf{R}^{2 \times n}$, and the initial state $x(1)$, are given.

The vehicle must reach a destination $d \in \mathbf{R}^2$ at time $t = M + N + 1$, where $M > 0$ and $N > 0$, *i.e.*, we must have $p(M + N + 1) = d$. The subtlety here is that we are not told what d is at $t = 1$; we simply know that it is one of K possible destinations $d^{(1)}, \ldots, d^{(K)}$ (which we are given). At time $t = M + 1$, the destination (which is one of $d^{(1)}, \ldots, d^{(K)}$) will be revealed to you.

Thus, you must choose the inputs up to time M, $u(1), u(2), \ldots, u(M)$, independent of the actual final destination; but you can choose $u(M + 1), \ldots, u(M + N)$ depending on the final destination. We will denote the choice of these inputs, in the case when the final destination is $d^{(k)}$, as $u^{(k)}(M + 1), \ldots, u^{(k)}(M + N)$.

We will choose the inputs to minimize the cost function

$$\sum_{t=1}^{M} \|u(t)\|^2 + \frac{1}{K} \sum_{k=1}^{K} \sum_{t=M+1}^{M+N} \|u^{(k)}(t)\|^2,$$

which is the sum of squared-norm costs, averaged over all destinations.

(a) Explain how to find $u(1), \ldots, u(M)$ and $u^{(k)}(M + 1), \ldots, u^{(k)}(M + N)$, for $k = 1, \ldots, K$.

(b) Carry out your method on the data given in `delayed_dest_data.m`. Report the optimal cost function value, and for each possible destination plot the position of the vehicle $p^{(k)}(1), \ldots, p^{(k)}(M + N + 1)$. (The data file contains commented-out code for producing your plots.)

Comment briefly on the following statement: "Since we do not know where we are supposed to go until $t = M + 1$, there's no point using the input (for which we are charged) until then."

5. *One of these vectors doesn't fit.* The file `one_of_these_data.m` contains an $n \times m$ matrix `X`, whose columns we denote as $x^{(1)}, \ldots, x^{(m)} \in \mathbf{R}^n$. The columns are (vector) data collected in some application. The ordering of the vectors isn't relevant; in other words, permuting the columns would make no difference.

   One of the vectors doesn't fit with the others.

   Find the index of the vector that doesn't fit. Carefully explain your method, and especially, in what way the vector you've chosen doesn't fit with the others. Your explanation can be algebraic, or geometric (or both), but it should be simple to state, and involve ideas and methods from this course.

   Since the question is vague, clarity in your explanation of your method and approach is very important. In particular, we want a nice, short explanation. *We will not read a long, complicated, or rambling explanation.*

6. *Tax policies.* In this problem we explore a dynamic model of an economy, including the effects of government taxes and spending, which we assume (for simplicity) takes place at the beginning of each year. Let $x(t) \in \mathbf{R}^n$ represent the pre-tax economic activity at the beginning of year t, across n sectors, with $x(t)_i$ being the pre-tax activity level in sector i. We let $\tilde{x}(t) \in \mathbf{R}^n$ denote the post-tax economic activity, across n sectors, at the beginning of year t. We will assume that all entries of $x(0)$ are positive, which will imply that all entries of $x(t)$ and $\tilde{x}(t)$ are positive, for all $t \geq 0$.

   The pre- and post-tax activity levels are related as follows. The government taxes the sector activities at rates given by $r \in \mathbf{R}^n$, with r_i the tax rate for sector i. These rates all satisfy $0 \leq r_i < 1$. The total government revenue is then $R(t) = r^T x(t)$. This total revenue is then spent in the sectors proportionally, with $s \in \mathbf{R}^n$ giving the spending proportions in the sectors. These spending proportions satisfy $s_i \geq 0$ and $\sum_{i=1}^n s_i = 1$; the spending in sector i is $s_i R(t)$. The post-tax economic activity in sector i, which accounts for the government taxes and spending, is then given by

$$\tilde{x}(t)_i = x(t)_i - r_i x(t)_i + s_i R(t), \quad i = 1, \ldots, n, \quad t = 0, 1, \ldots.$$

   Economic activity propagates from year to year as $x(t+1) = E\tilde{x}(t)$, where $E \in \mathbf{R}^{n \times n}$ is the input-output matrix of the economy. You can assume that all entries of E are positive.

   We let $S(t) = \sum_{i=1}^n x(t)_i$ denote the total economic activity in year t, and we let

$$G = \lim_{t \to \infty} \frac{S(t+1)}{S(t)}$$

   denote the (asymptotic) growth rate (assuming it exceeds one) of the economy.

   (a) Explain why the growth rate does not depend on $x(0)$ (unless it exactly satisfies a single linear equation, which we rule out as essentially impossible). Express the growth rate G in terms of the problem data r, s, and E, using ideas from the course. You may assume that a matrix that arises in your analysis is diagonalizable and has a single dominant eigenvalue, *i.e.*, an eigenvalue λ_1 that satisfies $|\lambda_1| > |\lambda_i|$ for $i = 2, \ldots, n$. (These assumptions aren't actually needed—they're just to simplify the problem for you.)

   (b) Consider the problem instance with data

$$E = \begin{bmatrix} 0.3 & 0.4 & 0.1 & 0.6 \\ 0.2 & 0.3 & 0.7 & 0.2 \\ 0.1 & 0.2 & 0.2 & 0.1 \\ 0.4 & 0.2 & 0.3 & 0.2 \end{bmatrix}, \quad r = \begin{bmatrix} 0.45 \\ 0.25 \\ 0.1 \\ 0.1 \end{bmatrix}, \quad s = \begin{bmatrix} 0.15 \\ 0.3 \\ 0.4 \\ 0.15 \end{bmatrix}.$$

   Find the growth rate. Now find the growth rate with the tax rate set to zero, *i.e.*, $r = 0$ (in which case s doesn't matter). You are welcome (even, encouraged) to simulate the economic activity to double-check your answer, but we want the value using the expression found in part (a).

7. *Regulation using ternary inputs.* Consider a discrete-time linear dynamical system

$$x(t+1) = Ax(t) + bu(t), \quad t = 1, 2, \ldots,$$

with $x(t) \in \mathbf{R}^n$, $u(t) \in \{-1, 0, 1\}$, and $b \in \mathbf{R}^n$, $b \neq 0$. (The problem title comes from the restriction that the input can only take three possible values.) Our goal is to regulate the system, *i.e.*, choose the inputs $u(t)$ so as to drive the state $x(t)$ towards zero. We will adopt a greedy strategy: At each time t, we will choose $u(t)$ so as to minimize $\|x(t+1)\|$.

(a) Show that $u(t)$ has the form $u(t) = \mathbf{round}(kx(t))$, where $k \in \mathbf{R}^{1\times n}$, and $\mathbf{round}(a)$ rounds the number a to the closest of $\{-1, 0, 1\}$, *i.e.*,

$$\mathbf{round}(a) = \begin{cases} 1 & a > 1/2 \\ 0 & |a| \leq 1/2 \\ -1 & a < -1/2. \end{cases}$$

(We don't care about what happens when there are ties; we have arbitrarily broken ties in favor of $a = 0$.) Give an explicit expression for k.

(b) Consider the specific problem with data

$$A = \begin{bmatrix} 1 & .2 & -.2 & 0 \\ -.2 & 1 & 0 & .15 \\ .2 & 0 & .9 & 0 \\ 0 & -.15 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} .1 \\ -.1 \\ .1 \\ -.1 \end{bmatrix}, \qquad x(1) = \begin{bmatrix} -4 \\ 0 \\ 0 \\ -4 \end{bmatrix}.$$

Give k, and plot $\|x(t)\|$ and $u(t)$ for $t = 1, \ldots, 100$. Use the matlab function `stairs` to plot $u(t)$.