

Lecture 13

Linear dynamical systems with inputs & outputs

- inputs & outputs: interpretations
- transfer function
- impulse and step responses
- examples

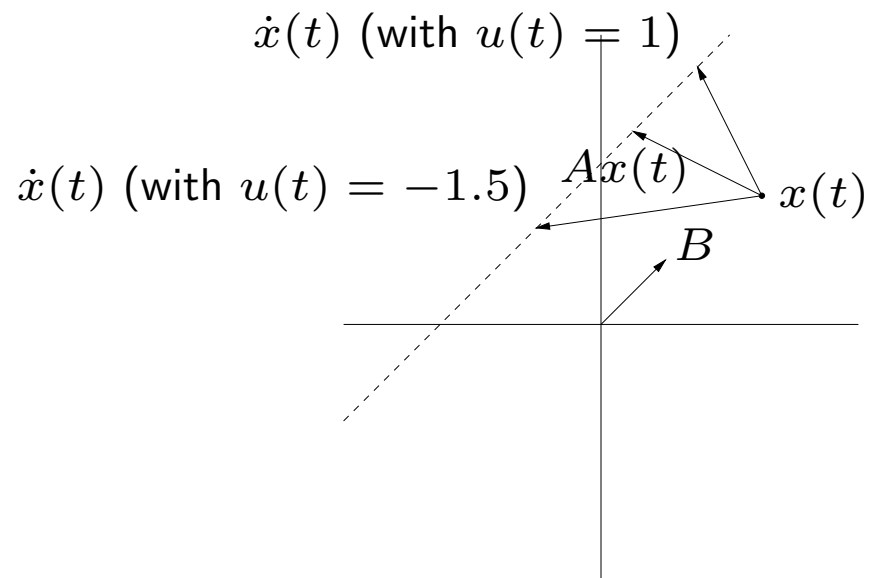
Inputs & outputs

recall continuous-time time-invariant LDS has form

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

- Ax is called the *drift term* (of \dot{x})
- Bu is called the *input term* (of \dot{x})

picture, with $B \in \mathbf{R}^{2 \times 1}$:



Interpretations

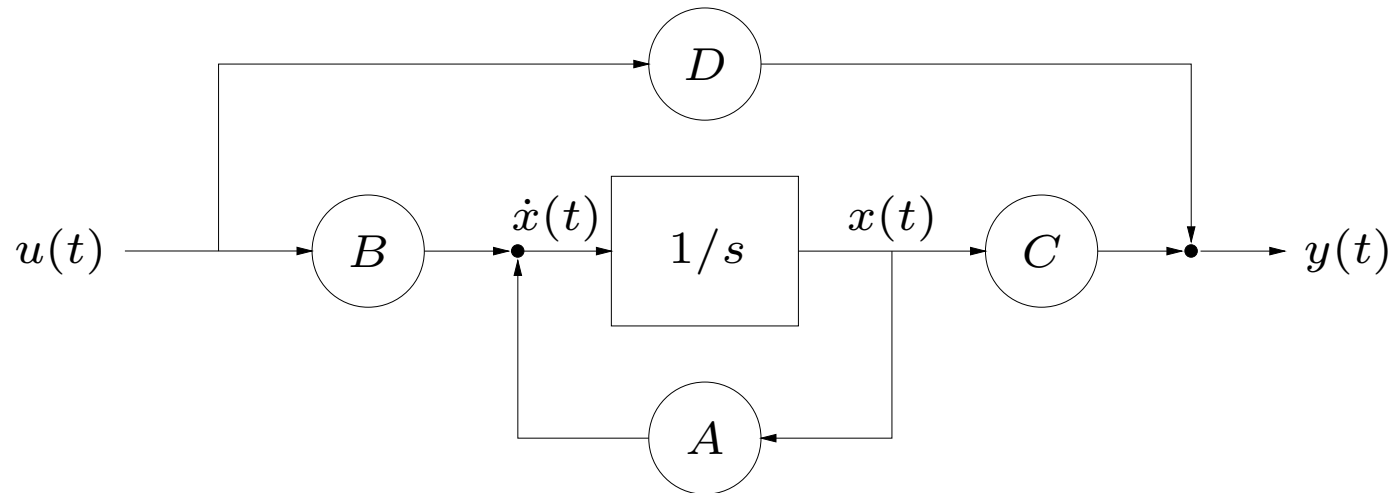
write $\dot{x} = Ax + b_1u_1 + \cdots + b_mu_m$, where $B = [b_1 \cdots b_m]$

- state derivative is sum of autonomous term (Ax) and one term per input (b_iu_i)
- each input u_i gives another degree of freedom for \dot{x} (assuming columns of B independent)

write $\dot{x} = Ax + Bu$ as $\dot{x}_i = \tilde{a}_i^T x + \tilde{b}_i^T u$, where $\tilde{a}_i^T, \tilde{b}_i^T$ are the rows of A, B

- i th state derivative is linear function of state x and input u

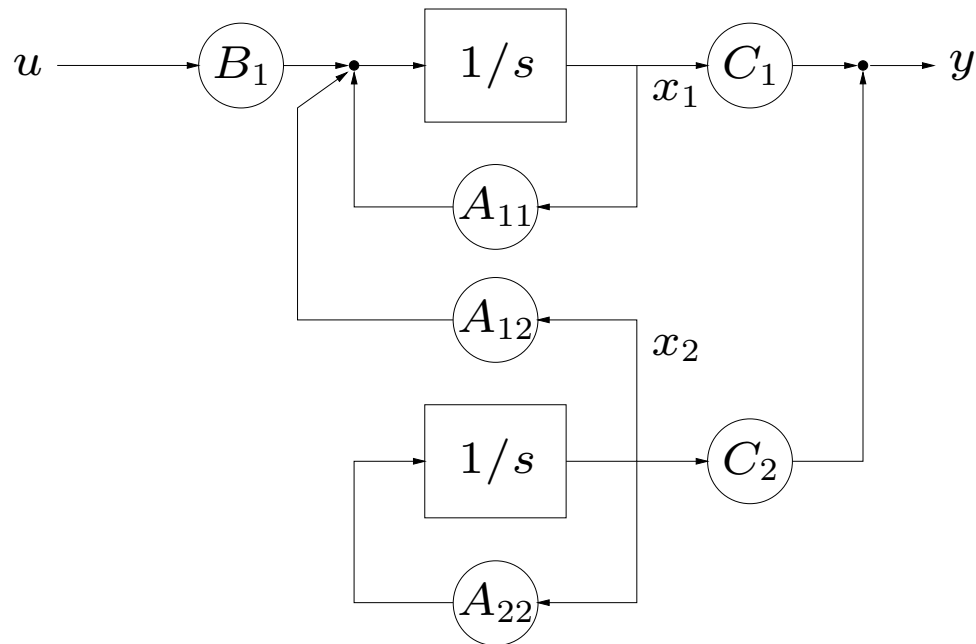
Block diagram



- A_{ij} is gain factor from state x_j into integrator i
- B_{ij} is gain factor from input u_j into integrator i
- C_{ij} is gain factor from state x_j into output y_i
- D_{ij} is gain factor from input u_j into output y_i

interesting when there is structure, *e.g.*, with $x_1 \in \mathbf{R}^{n_1}$, $x_2 \in \mathbf{R}^{n_2}$:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



- x_2 is not affected by input u , *i.e.*, x_2 propagates autonomously
- x_2 affects y directly and through x_1

Transfer function

take Laplace transform of $\dot{x} = Ax + Bu$:

$$sX(s) - x(0) = AX(s) + BU(s)$$

hence

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

so

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau) d\tau$$

- $e^{tA}x(0)$ is the unforced or autonomous response
- $e^{tA}B$ is called the input-to-state impulse response or impulse matrix
- $(sI - A)^{-1}B$ is called the *input-to-state transfer function* or *transfer matrix*

with $y = Cx + Du$ we have:

$$Y(s) = C(sI - A)^{-1}x(0) + (C(sI - A)^{-1}B + D)U(s)$$

so

$$y(t) = Ce^{tA}x(0) + \int_0^t Ce^{(t-\tau)A}Bu(\tau) d\tau + Du(t)$$

- output term $Ce^{tA}x(0)$ due to initial condition
- $H(s) = C(sI - A)^{-1}B + D$ is called the *transfer function* or *transfer matrix*
- $h(t) = Ce^{tA}B + D\delta(t)$ is called the *impulse response* or *impulse matrix* (δ is the Dirac delta function)

with zero initial condition we have:

$$Y(s) = H(s)U(s), \quad y = h * u$$

where $*$ is convolution (of matrix valued functions)

intepretation:

- H_{ij} is transfer function from input u_j to output y_i

Impulse response

impulse response $h(t) = Ce^{tA}B + D\delta(t)$

with $x(0) = 0$, $y = h * u$, *i.e.*,

$$y_i(t) = \sum_{j=1}^m \int_0^t h_{ij}(t - \tau) u_j(\tau) d\tau$$

interpretations:

- $h_{ij}(t)$ is impulse response from j th input to i th output
- $h_{ij}(t)$ gives y_i when $u(t) = e_j\delta$
- $h_{ij}(\tau)$ shows how dependent output i is, on what input j was, τ seconds ago
- i indexes output; j indexes input; τ indexes time lag

Step response

the *step response* or *step matrix* is given by

$$s(t) = \int_0^t h(\tau) d\tau$$

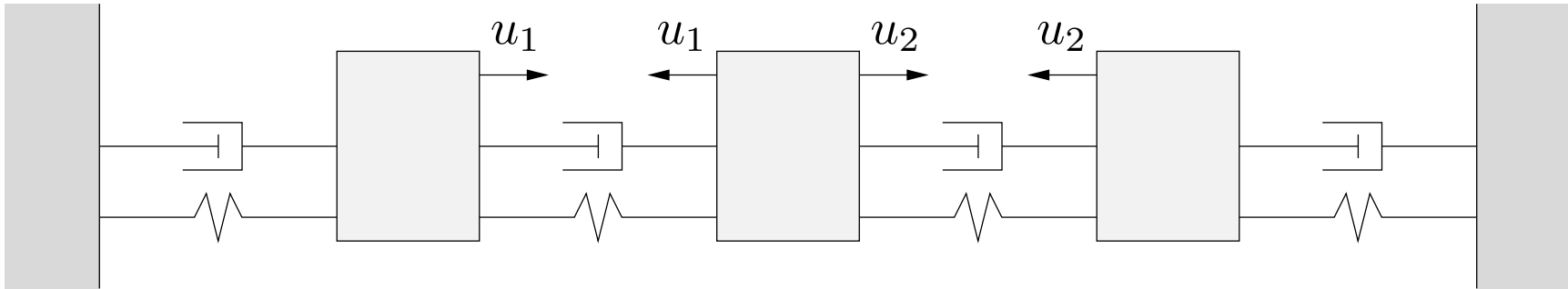
interpretations:

- $s_{ij}(t)$ is step response from j th input to i th output
- $s_{ij}(t)$ gives y_i when $u = e_j$ for $t \geq 0$

for invertible A , we have

$$s(t) = CA^{-1} (e^{tA} - I) B + D$$

Example 1



- unit masses, springs, dampers
- u_1 is tension between 1st & 2nd masses
- u_2 is tension between 2nd & 3rd masses
- $y \in \mathbf{R}^3$ is displacement of masses 1,2,3
- $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

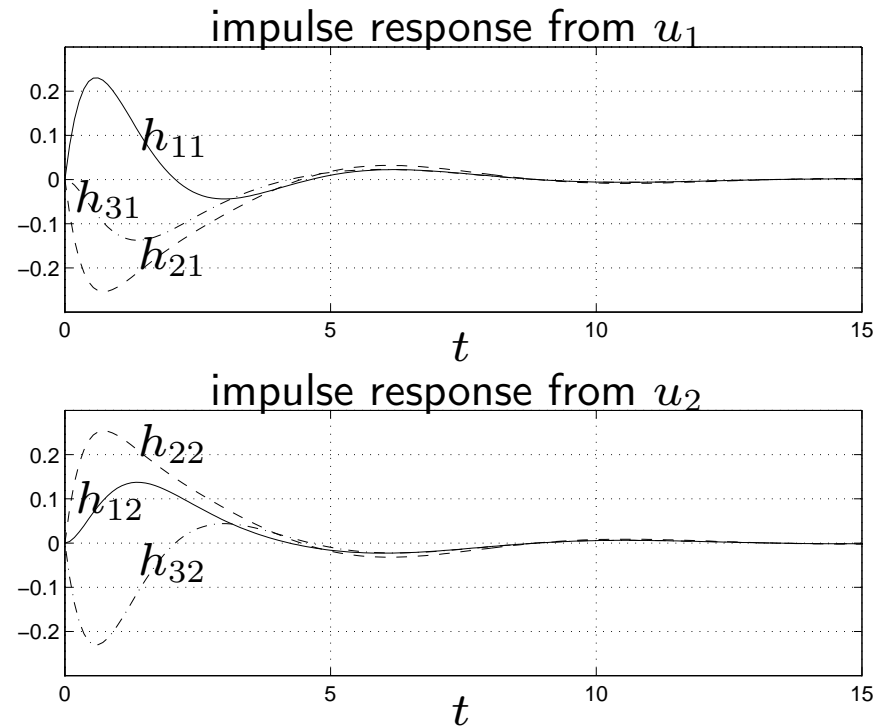
system is:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -2 & 1 & 0 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 0 & 1 & -2 & 0 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

eigenvalues of A are

$$-1.71 \pm j0.71, \quad -1.00 \pm j1.00, \quad -0.29 \pm j0.71$$

impulse response:

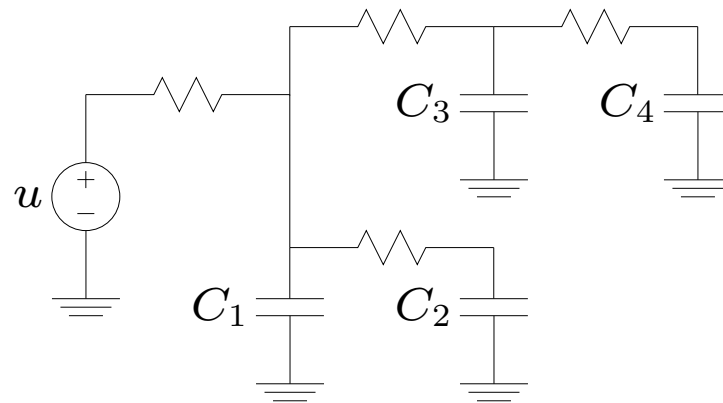


roughly speaking:

- impulse at u_1 affects third mass less than other two
- impulse at u_2 affects first mass later than other two

Example 2

interconnect circuit:



- $u(t) \in \mathbf{R}$ is input (drive) voltage
- x_i is voltage across C_i
- output is state: $y = x$
- unit resistors, unit capacitors
- step response matrix shows delay to each node

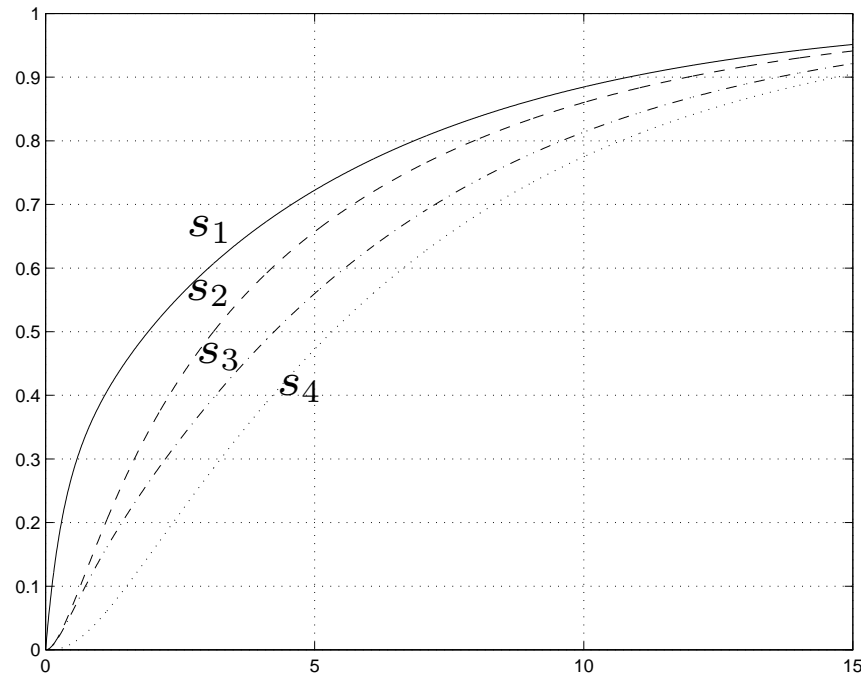
system is

$$\dot{x} = \begin{bmatrix} -3 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u, \quad y = x$$

eigenvalues of A are

$$-0.17, \quad -0.66, \quad -2.21, \quad -3.96$$

step response matrix $s(t) \in \mathbf{R}^{4 \times 1}$:



- shortest delay to x_1 ; longest delay to x_4
- delays ≈ 10 , consistent with slowest (*i.e.*, dominant) eigenvalue -0.17

DC or static gain matrix

- transfer function at $s = 0$ is $H(0) = -CA^{-1}B + D \in \mathbf{R}^{m \times p}$
- DC transfer function describes system under *static* conditions, *i.e.*, x , u , y constant:

$$0 = \dot{x} = Ax + Bu, \quad y = Cx + Du$$

eliminate x to get $y = H(0)u$

- if system is stable,

$$H(0) = \int_0^{\infty} h(t) dt = \lim_{t \rightarrow \infty} s(t)$$

$$\text{(recall: } H(s) = \int_0^{\infty} e^{-st} h(t) dt, s(t) = \int_0^t h(\tau) d\tau)$$

if $u(t) \rightarrow u_{\infty} \in \mathbf{R}^m$, then $y(t) \rightarrow y_{\infty} \in \mathbf{R}^p$ where $y_{\infty} = H(0)u_{\infty}$

DC gain matrix for example 1 (springs):

$$H(0) = \begin{bmatrix} 1/4 & 1/4 \\ -1/2 & 1/2 \\ -1/4 & -1/4 \end{bmatrix}$$

DC gain matrix for example 2 (RC circuit):

$$H(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(do these make sense?)

Discretization with piecewise constant inputs

linear system $\dot{x} = Ax + Bu, y = Cx + Du$

suppose $u_d : \mathbf{Z}_+ \rightarrow \mathbf{R}^m$ is a sequence, and

$$u(t) = u_d(k) \quad \text{for } kh \leq t < (k+1)h, \quad k = 0, 1, \dots$$

define sequences

$$x_d(k) = x(kh), \quad y_d(k) = y(kh), \quad k = 0, 1, \dots$$

- $h > 0$ is called the *sample interval* (for x and y) or *update interval* (for u)
- u is piecewise constant (called *zero-order-hold*)
- x_d, y_d are sampled versions of x, y

$$\begin{aligned}
x_d(k+1) &= x((k+1)h) \\
&= e^{hA}x(kh) + \int_0^h e^{\tau A}Bu((k+1)h - \tau) d\tau \\
&= e^{hA}x_d(k) + \left(\int_0^h e^{\tau A} d\tau \right) B u_d(k)
\end{aligned}$$

x_d , u_d , and y_d satisfy discrete-time LDS equations

$$x_d(k+1) = A_d x_d(k) + B_d u_d(k), \quad y_d(k) = C_d x_d(k) + D_d u_d(k)$$

where

$$A_d = e^{hA}, \quad B_d = \left(\int_0^h e^{\tau A} d\tau \right) B, \quad C_d = C, \quad D_d = D$$

called *discretized system*

if A is invertible, we can express integral as

$$\int_0^h e^{\tau A} d\tau = A^{-1} (e^{hA} - I)$$

stability: if eigenvalues of A are $\lambda_1, \dots, \lambda_n$, then eigenvalues of A_d are $e^{h\lambda_1}, \dots, e^{h\lambda_n}$

discretization preserves stability properties since

$$\Re \lambda_i < 0 \quad \Leftrightarrow \quad |e^{h\lambda_i}| < 1$$

for $h > 0$

extensions/variations:

- *offsets*: updates for u and sampling of x, y are offset in time
- *multirate*: u_i updated, y_i sampled at different intervals
(usually integer multiples of a common interval h)

both very common in practice

Dual system

the *dual system* associated with system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

is given by

$$\dot{z} = A^T z + C^T v, \quad w = B^T z + D^T v$$

- all matrices are transposed
- role of B and C are swapped

transfer function of dual system:

$$(B^T)(sI - A^T)^{-1}(C^T) + D^T = H(s)^T$$

where $H(s) = C(sI - A)^{-1}B + D$

(for SISO case, TF of dual is same as original)

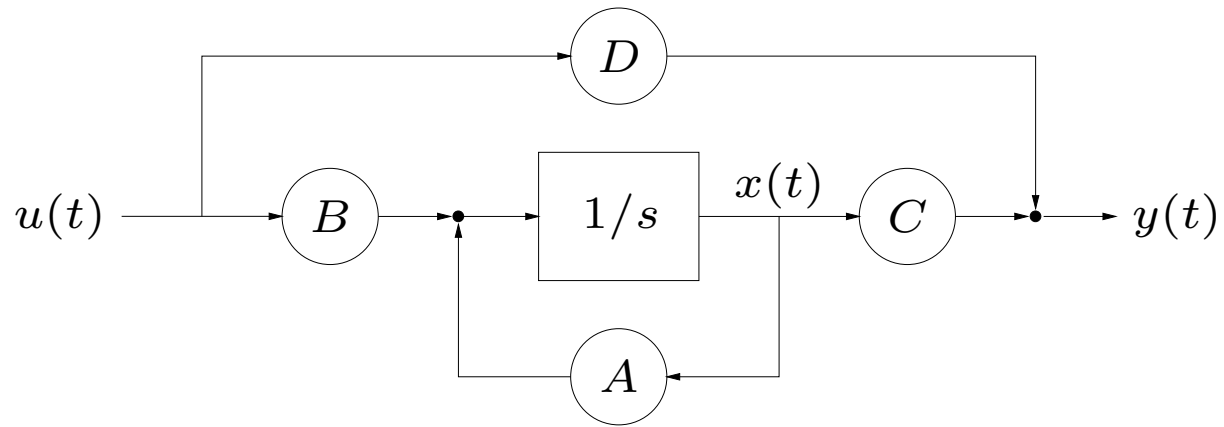
eigenvalues (hence stability properties) are the same

Dual via block diagram

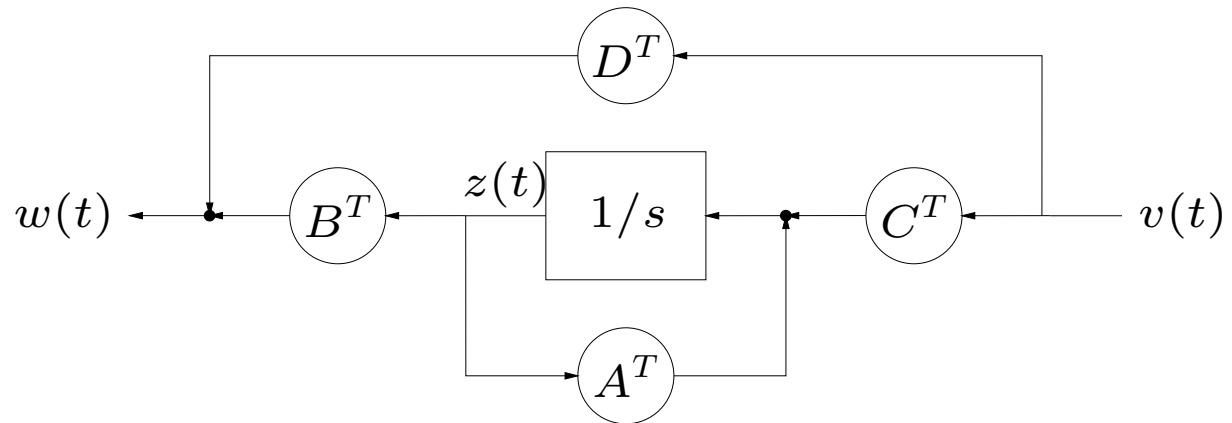
in terms of block diagrams, dual is formed by:

- transpose all matrices
- swap inputs and outputs on all boxes
- reverse directions of signal flow arrows
- swap solder joints and summing junctions

original system:



dual system:



Causality

interpretation of

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau) d\tau$$

$$y(t) = Ce^{tA}x(0) + \int_0^t Ce^{(t-\tau)A}Bu(\tau) d\tau + Du(t)$$

for $t \geq 0$:

current state ($x(t)$) and output ($y(t)$) depend on *past* input ($u(\tau)$ for $\tau \leq t$)

i.e., mapping from input to state and output is *causal* (with fixed *initial* state)

now consider fixed *final* state $x(T)$: for $t \leq T$,

$$x(t) = e^{(t-T)A}x(T) + \int_T^t e^{(t-\tau)A}Bu(\tau) d\tau,$$

i.e., current state (and output) depend on future input!

so for fixed final condition, same system is anti-causal

Idea of state

$x(t)$ is called *state* of system at time t since:

- future output depends only on current state and future input
- future output depends on past input only through current state
- state summarizes effect of past inputs on future output
- state is bridge between past inputs and future outputs

Change of coordinates

start with LDS $\dot{x} = Ax + Bu$, $y = Cx + Du$

change coordinates in \mathbf{R}^n to \tilde{x} , with $x = T\tilde{x}$

then

$$\dot{\tilde{x}} = T^{-1}\dot{x} = T^{-1}(Ax + Bu) = T^{-1}AT\tilde{x} + T^{-1}Bu$$

hence LDS can be expressed as

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \quad y = \tilde{C}\tilde{x} + \tilde{D}u$$

where

$$\tilde{A} = T^{-1}AT, \quad \tilde{B} = T^{-1}B, \quad \tilde{C} = CT, \quad \tilde{D} = D$$

TF is same (since u , y aren't affected):

$$\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D} = C(sI - A)^{-1}B + D$$

Standard forms for LDS

can change coordinates to put A in various forms (diagonal, real modal, Jordan . . .)

e.g., to put LDS in *diagonal form*, find T s.t.

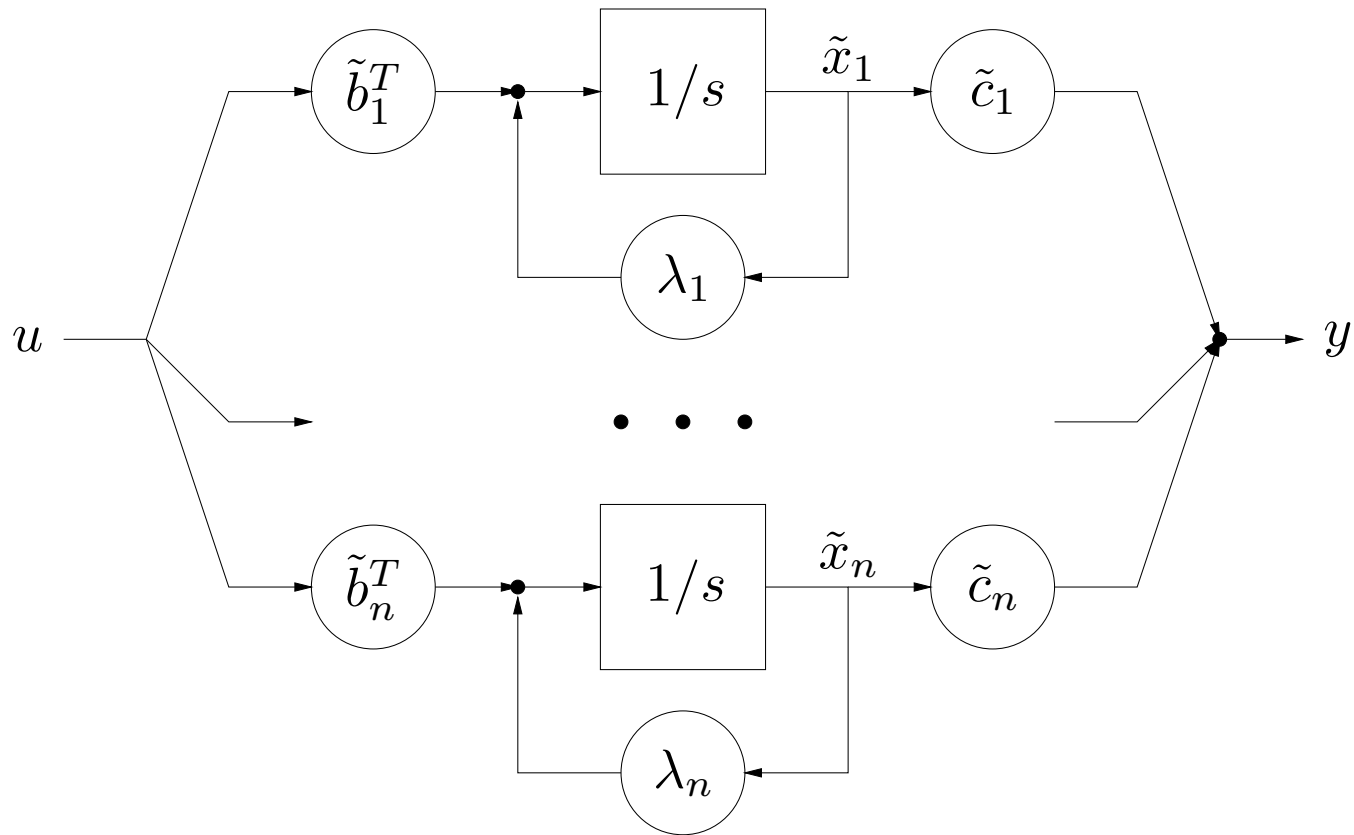
$$T^{-1}AT = \mathbf{diag}(\lambda_1, \dots, \lambda_n)$$

write

$$T^{-1}B = \begin{bmatrix} \tilde{b}_1^T \\ \vdots \\ \tilde{b}_n^T \end{bmatrix}, \quad CT = [\tilde{c}_1 \quad \cdots \quad \tilde{c}_n]$$

so

$$\dot{\tilde{x}}_i = \lambda_i \tilde{x}_i + \tilde{b}_i^T u, \quad y = \sum_{i=1}^n \tilde{c}_i \tilde{x}_i$$

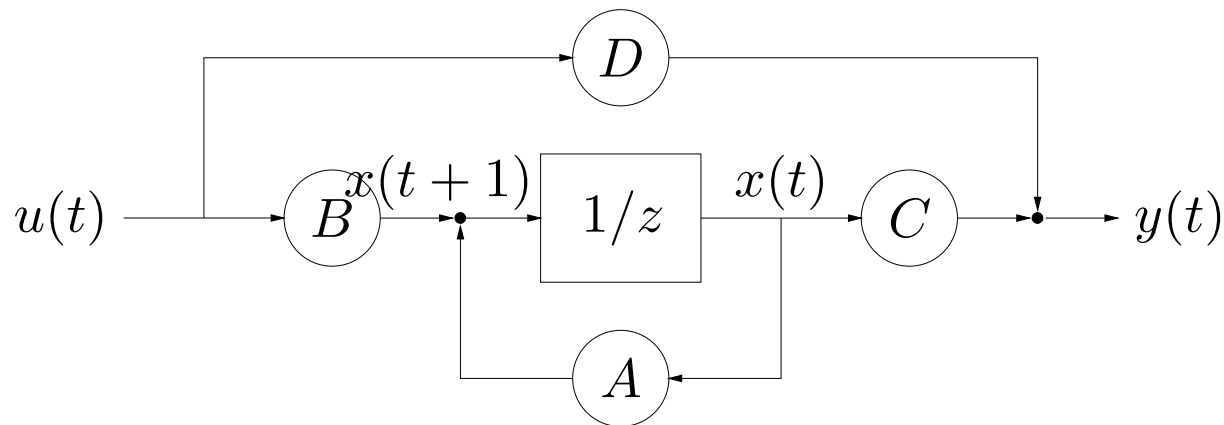


(here we assume $D = 0$)

Discrete-time systems

discrete-time LDS:

$$x(t + 1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$



- only difference w/cts-time: z instead of s
- interpretation of z^{-1} block:
 - unit delayor (shifts sequence back in time one epoch)
 - latch (plus small delay to avoid race condition)

we have:

$$x(1) = Ax(0) + Bu(0),$$

$$\begin{aligned} x(2) &= Ax(1) + Bu(1) \\ &= A^2x(0) + ABu(0) + Bu(1), \end{aligned}$$

and in general, for $t \in \mathbf{Z}_+$,

$$x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{(t-1-\tau)} Bu(\tau)$$

hence

$$y(t) = CA^t x(0) + h * u$$

where $*$ is discrete-time convolution and

$$h(t) = \begin{cases} D, & t = 0 \\ CA^{t-1}B, & t > 0 \end{cases}$$

is the impulse response

\mathcal{Z} -transform

suppose $w \in \mathbf{R}^{p \times q}$ is a sequence (discrete-time signal), *i.e.*,

$$w : \mathbf{Z}_+ \rightarrow \mathbf{R}^{p \times q}$$

recall \mathcal{Z} -transform $W = \mathcal{Z}(w)$:

$$W(z) = \sum_{t=0}^{\infty} z^{-t} w(t)$$

where $W : D \subseteq \mathbf{C} \rightarrow \mathbf{C}^{p \times q}$ (D is domain of W)

time-advanced or shifted signal v :

$$v(t) = w(t + 1), \quad t = 0, 1, \dots$$

\mathcal{Z} -transform of time-advanced signal:

$$\begin{aligned} V(z) &= \sum_{t=0}^{\infty} z^{-t} w(t+1) \\ &= z \sum_{t=1}^{\infty} z^{-t} w(t) \\ &= zW(z) - zw(0) \end{aligned}$$

Discrete-time transfer function

take \mathcal{Z} -transform of system equations

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

yields

$$zX(z) - zx(0) = AX(z) + BU(z), \quad Y(z) = CX(z) + DU(z)$$

solve for $X(z)$ to get

$$X(z) = (zI - A)^{-1}zx(0) + (zI - A)^{-1}BU(z)$$

(note extra z in first term!)

hence

$$Y(z) = H(z)U(z) + C(zI - A)^{-1}zx(0)$$

where $H(z) = C(zI - A)^{-1}B + D$ is the *discrete-time transfer function*

note power series expansion of resolvent:

$$(zI - A)^{-1} = z^{-1}I + z^{-2}A + z^{-3}A^2 + \dots$$