

# Lecture 8

## Least-norm solutions of undetermined equations

- least-norm solution of underdetermined equations
- minimum norm solutions via  $QR$  factorization
- derivation via Lagrange multipliers
- relation to regularized least-squares
- general norm minimization with equality constraints

# Underdetermined linear equations

we consider

$$y = Ax$$

where  $A \in \mathbf{R}^{m \times n}$  is fat ( $m < n$ ), *i.e.*,

- there are more variables than equations
- $x$  is *underspecified*, *i.e.*, many choices of  $x$  lead to the same  $y$

we'll assume that  $A$  is full rank ( $m$ ), so for each  $y \in \mathbf{R}^m$ , there is a solution set of all solutions has form

$$\{ x \mid Ax = y \} = \{ x_p + z \mid z \in \mathcal{N}(A) \}$$

where  $x_p$  is any ('particular') solution, *i.e.*,  $Ax_p = y$

- $z$  characterizes available choices in solution
- solution has  $\dim \mathcal{N}(A) = n - m$  'degrees of freedom'
- can choose  $z$  to satisfy other specs or optimize among solutions

# Least-norm solution

one particular solution is

$$x_{\text{ln}} = A^T (AA^T)^{-1} y$$

( $AA^T$  is invertible since  $A$  full rank)

in fact,  $x_{\text{ln}}$  is the solution of  $y = Ax$  that minimizes  $\|x\|$

*i.e.*,  $x_{\text{ln}}$  is solution of optimization problem

$$\begin{array}{ll} \text{minimize} & \|x\| \\ \text{subject to} & Ax = y \end{array}$$

(with variable  $x \in \mathbf{R}^n$ )

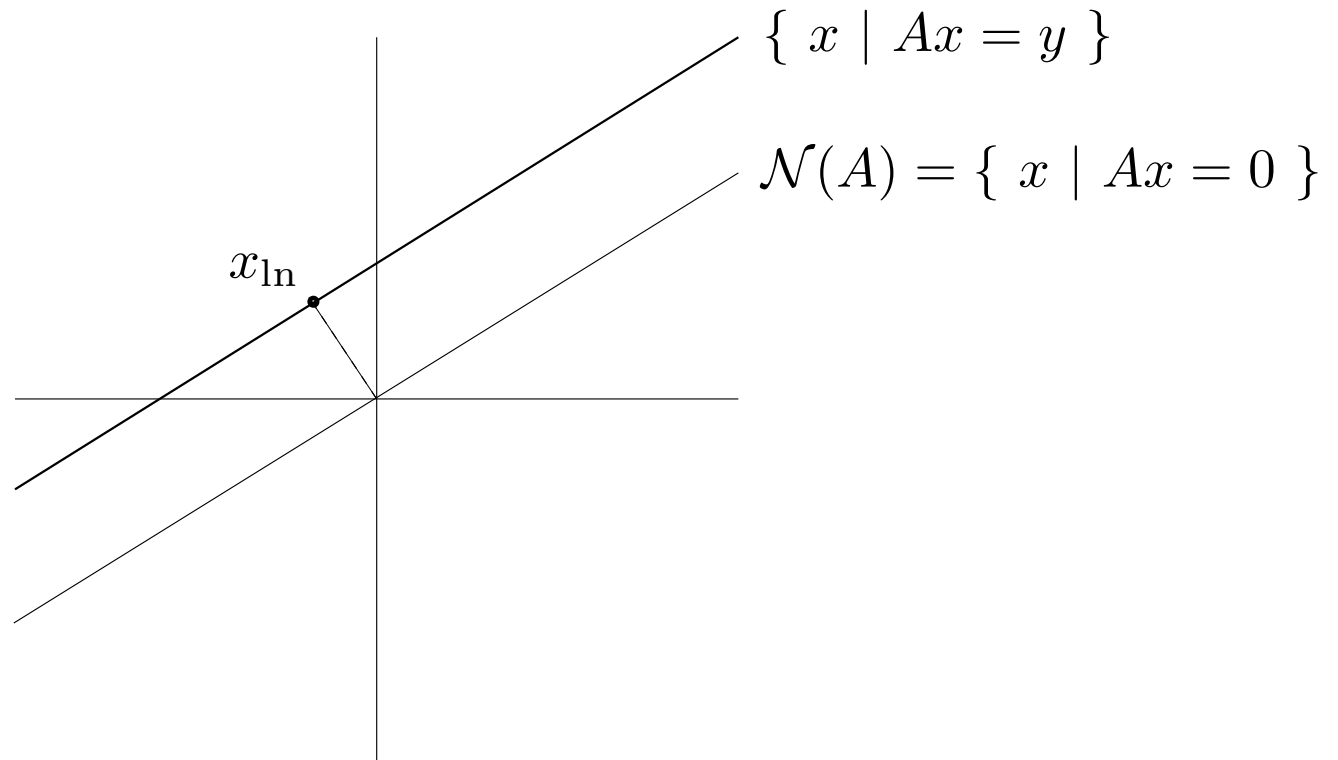
suppose  $Ax = y$ , so  $A(x - x_{\text{ln}}) = 0$  and

$$\begin{aligned}(x - x_{\text{ln}})^T x_{\text{ln}} &= (x - x_{\text{ln}})^T A^T (AA^T)^{-1} y \\ &= (A(x - x_{\text{ln}}))^T (AA^T)^{-1} y \\ &= 0\end{aligned}$$

*i.e.*,  $(x - x_{\text{ln}}) \perp x_{\text{ln}}$ , so

$$\|x\|^2 = \|x_{\text{ln}} + x - x_{\text{ln}}\|^2 = \|x_{\text{ln}}\|^2 + \|x - x_{\text{ln}}\|^2 \geq \|x_{\text{ln}}\|^2$$

*i.e.*,  $x_{\text{ln}}$  has smallest norm of any solution



- **orthogonality condition:**  $x_{ln} \perp \mathcal{N}(A)$
- **projection interpretation:**  $x_{ln}$  is projection of 0 on solution set  $\{ x \mid Ax = y \}$

- $A^\dagger = A^T(AA^T)^{-1}$  is called the *pseudo-inverse* of full rank, fat  $A$
- $A^T(AA^T)^{-1}$  is a *right inverse* of  $A$
- $I - A^T(AA^T)^{-1}A$  gives projection onto  $\mathcal{N}(A)$

cf. analogous formulas for full rank, **skinny** matrix  $A$ :

- $A^\dagger = (A^T A)^{-1}A^T$
- $(A^T A)^{-1}A^T$  is a *left inverse* of  $A$
- $A(A^T A)^{-1}A^T$  gives projection onto  $\mathcal{R}(A)$

## Least-norm solution via QR factorization

find  $QR$  factorization of  $A^T$ , *i.e.*,  $A^T = QR$ , with

- $Q \in \mathbf{R}^{n \times m}$ ,  $Q^T Q = I_m$
- $R \in \mathbf{R}^{m \times m}$  upper triangular, nonsingular

then

- $x_{\text{ln}} = A^T (AA^T)^{-1} y = QR^{-T} y$
- $\|x_{\text{ln}}\| = \|R^{-T} y\|$

# Derivation via Lagrange multipliers

- least-norm solution solves optimization problem

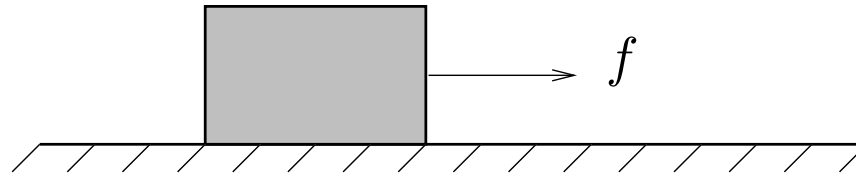
$$\begin{array}{ll} \text{minimize} & x^T x \\ \text{subject to} & Ax = y \end{array}$$

- introduce Lagrange multipliers:  $L(x, \lambda) = x^T x + \lambda^T (Ax - y)$
- optimality conditions are

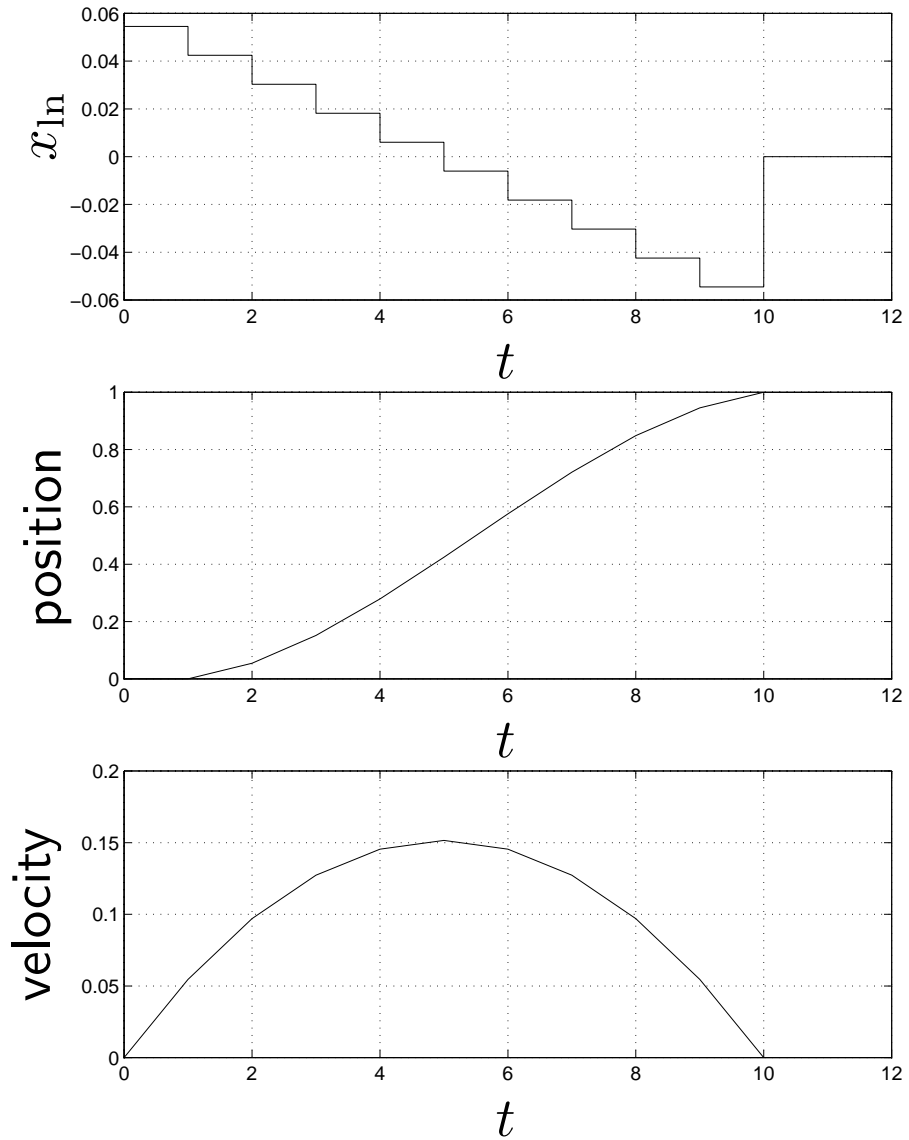
$$\nabla_x L = 2x + A^T \lambda = 0, \quad \nabla_\lambda L = Ax - y = 0$$

- from first condition,  $x = -A^T \lambda / 2$
- substitute into second to get  $\lambda = -2(AA^T)^{-1}y$
- hence  $x = A^T (AA^T)^{-1}y$

## Example: transferring mass unit distance



- unit mass at rest subject to forces  $x_i$  for  $i - 1 < t \leq i$ ,  $i = 1, \dots, 10$
- $y_1$  is position at  $t = 10$ ,  $y_2$  is velocity at  $t = 10$
- $y = Ax$  where  $A \in \mathbf{R}^{2 \times 10}$  ( $A$  is fat)
- find least norm force that transfers mass unit distance with zero final velocity, *i.e.*,  $y = (1, 0)$



## Relation to regularized least-squares

- suppose  $A \in \mathbf{R}^{m \times n}$  is fat, full rank
- define  $J_1 = \|Ax - y\|^2$ ,  $J_2 = \|x\|^2$
- least-norm solution minimizes  $J_2$  with  $J_1 = 0$
- minimizer of weighted-sum objective  $J_1 + \mu J_2 = \|Ax - y\|^2 + \mu\|x\|^2$  is

$$x_\mu = (A^T A + \mu I)^{-1} A^T y$$

- **fact:**  $x_\mu \rightarrow x_{\text{ln}}$  as  $\mu \rightarrow 0$ , *i.e.*, regularized solution converges to least-norm solution as  $\mu \rightarrow 0$
- in matrix terms: as  $\mu \rightarrow 0$ ,

$$(A^T A + \mu I)^{-1} A^T \rightarrow A^T (A A^T)^{-1}$$

(for full rank, fat  $A$ )

# General norm minimization with equality constraints

consider problem

$$\begin{array}{ll} \text{minimize} & \|Ax - b\| \\ \text{subject to} & Cx = d \end{array}$$

with variable  $x$

- includes least-squares and least-norm problems as special cases

- equivalent to

$$\begin{array}{ll} \text{minimize} & (1/2)\|Ax - b\|^2 \\ \text{subject to} & Cx = d \end{array}$$

- Lagrangian is

$$\begin{aligned} L(x, \lambda) &= (1/2)\|Ax - b\|^2 + \lambda^T(Cx - d) \\ &= (1/2)x^T A^T A x - b^T A x + (1/2)b^T b + \lambda^T C x - \lambda^T d \end{aligned}$$

- optimality conditions are

$$\nabla_x L = A^T Ax - A^T b + C^T \lambda = 0, \quad \nabla_\lambda L = Cx - d = 0$$

- write in block matrix form as

$$\begin{bmatrix} A^T A & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} A^T b \\ d \end{bmatrix}$$

- if the block matrix is invertible, we have

$$\begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} A^T A & C^T \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} A^T b \\ d \end{bmatrix}$$

if  $A^T A$  is invertible, we can derive a more explicit (and complicated) formula for  $x$

- from first block equation we get

$$x = (A^T A)^{-1}(A^T b - C^T \lambda)$$

- substitute into  $Cx = d$  to get

$$C(A^T A)^{-1}(A^T b - C^T \lambda) = d$$

so

$$\lambda = (C(A^T A)^{-1}C^T)^{-1} (C(A^T A)^{-1}A^T b - d)$$

- recover  $x$  from equation above (not pretty)

$$x = (A^T A)^{-1} \left( A^T b - C^T (C(A^T A)^{-1}C^T)^{-1} (C(A^T A)^{-1}A^T b - d) \right)$$